ACCURACY OF THE FOURIER EXTENSION METHOD FOR OSCILLATORY PHENOMENA

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Abstract

We introduce and analyse the so-called Fourier extension method for the approximation of oscillatory phenomena in bounded intervals. As we show, this method possesses good resolution properties for such problems. In particular, the resolution constant, the number of degrees of freedom per wavelength required to resolve an oscillation at a given frequency, can be varied between 2 and π by a user-determined parameter. The former value is optimal and achieved by Fourier series, but only when oscillations are periodic. Conversely, the Fourier extension method allows for the resolution of both periodic and nonperiodic oscillations with near-optimal complexity.

1 Introduction

In many physical problems one encounters the phenomenon of oscillation. When computing the solution to such a problem with a numerical method, this naturally leads to the question of resolution. That is, for a given frequency ω, how many degrees of freedom are required to resolve the solution to a desired accuracy [6, §3]? Whilst it may be impossible to answer this question in general, important *a priori* knowledge can be gained by studying a particular model class of problem [3, chpt. 2]. For example, in the unit interval $[-1, 1]$ (the case we consider in this paper), one typically considers the complex exponentials

$$f(x) = \exp(\mathrm{i}\pi\omega x), \quad (1.1)$$

where $\omega \in \mathbb{R}$ is the frequency of oscillation.

Whenever a problem has periodic oscillations, Fourier series present arguably the most effective means of approximation. In this case, as is easily seen from the model function (1.1), one witnesses a *resolution constant* of precisely 2 degrees of freedom per wavelength. In fact, this figure (referred to as the Nyquist rate in information theory) is optimal. Hence, Fourier series possess good *resolution power* for periodic oscillations.

The situation changes completely in the nonperiodic setting, in which case the Fourier approximation suffers from the Gibbs phenomenon and no longer converges uniformly on $[-1, 1]$. Put simply, Fourier series cannot be used to resolve nonperiodic oscillations.

A common alternative in this case is to expand f in orthogonal polynomials – Chebyshev or Legendre polynomials, for example. Whilst the orthogonal polynomial expansion of an analytic but nonperiodic function converges rapidly (in fact, exponentially fast), this approach leads to a suboptimal resolution constant of π [6, §3]. In Figure 1 we illustrate this behaviour by plotting the Chebyshev and Legendre expansion coefficients of (1.1). Note that the asymptotic decay of the n^{th} coefficient, exponentially fast in n, only sets in once n exceeds $\pi\omega$.

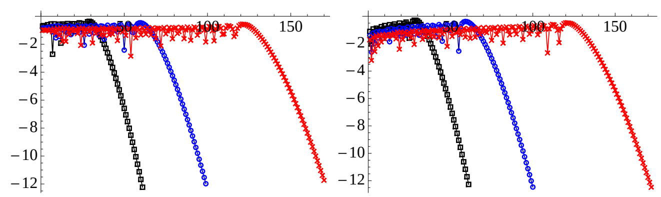

Figure 1: The values $\log_{10}|a_n|$ against $n = 1, 2, \ldots$, where a_n is the n^{th} Chebyshev (left) or Legendre (right) coefficient of (1.1) and $\omega = 10, 20, 40$.

A collection of commonly used alternative methods arises from the desire to reconstruct a function directly from its Fourier coefficients. Methods such as the Gegenbauer reconstruction technique [7], for example, typically offer spectral convergence, but with a significant deterioration in the resolution power [8].

Given this shortfall, the purpose of this paper is to study an alternative approach for oscillatory problems, the so-called *Fourier extension* method. As we discuss, this method involves a user-determined parameter $T \in (1, \infty)$, which allows the resolution constant $r = r(T)$ to be varied from 2 (in the limit $T \to 1$) to π ($T \to \infty$), with exponential convergence occurring once the number of degrees of freedom exceeds $r(T)\omega$. Hence, Fourier extensions are ideally suited for oscillatory problems.

Fourier extensions have been employed to overcome the Gibbs phenomenon in standard Fourier expansions [2]. Their application to surface parametrisations was explored in [4]. More recently, a method was developed in [5] to solve time-dependent PDEs in complex geometries. This was based on a similar, but not identical way of obtaining one-dimensional Fourier extensions, in combination with an alternating direction technique to handle general domains. Interestingly, it was shown in [10] (see also [5]) that this method leads to an absence of dispersion errors (or pollution errors) – another beneficial property for oscillatory wave problems shared with classical Fourier series, and very much related to resolution power.

In the next section, we introduce Fourier extensions and discuss their convergence. Resolution power is addressed in §3, and §4 contains numerical examples.

2 The Fourier extension method

Suppose that f is analytic and nonperiodic on $[-1, 1]$. One potential route towards approximating f to high accuracy is to seek an extension of f that is periodic on the extended domain $[-T, T]$, where $T > 1$ is an arbitrary parameter, and compute the Fourier series on $[-T, T]$ of this extension. If G_n is the space of functions of the form

$$g(x) = a_0 + \sum_{k=1}^{n} \left[a_k \cos \tfrac{k\pi}{T} x + b_k \sin \tfrac{k\pi}{T} x \right],$$

then a simple means for computing such an extension $g_n \in G_n$ is via the optimization problem

$$g_n := \arg \min_{g \in G_n} \|f - g\|_{\mathrm{L}^2[-1,1]}. \tag{2.1}$$

This leads to the *Fourier extension* method [2], [4], [9].

2.1 Convergence

Convergence of the Fourier extension method was considered in [9] for $T = 2$, and generalised to arbitrary $T > 1$ in [1]. For brevity, we shall only sketch details. Full proofs can be found in [1], [9].

We commence with the following:

Theorem 2.1 ([1]) *Suppose that $f \in \mathrm{H}^k[-1, 1]$ for some $k \in \mathbb{N}$, where $\mathrm{H}^k[-1, 1]$ is the k^{th} standard Sobolev space, and let $T_0 > 1$. Then, for all $T \geq T_0$, we have*

$$\|f - g_n\|_{\mathrm{L}^2[-1,1]} \leq c_k(T_0)(n\pi T^{-1})^{-k} \|f\|_{\mathrm{H}^k[-1,1]},$$

where g_n is the Fourier extension of f on $[-T, T]$.

This theorem confirms so-called spectral convergence (i.e. faster than any algebraic power of n^{-1}) of the Fourier extension method. However, whenever f is sufficiently regular, it turns out that the convergence rate is truly exponential – an observation first confirmed in [9] for $T = 2$. A generalisation of this result to arbitrary $T > 1$ is found in [1]. We now summarise the key details of the proof.

We first write $f(x) = f_e(x) + f_o(x)$ as a sum of the even and odd functions $f_e(x) = \frac{1}{2}(f(x) + f(-x))$ and $f_o(x) = \frac{1}{2}(f(x) - f(-x))$. Likewise, the Fourier extension g_n of f can also be decomposed into even and odd parts $g_n = g_{e,n} + g_{o,n}$, which may be considered separately. By symmetry, we restrict x to the subinterval $[0, 1]$.

Note that $g_{e,n} \in C_n$ and $g_{o,n} \in S_n$, where C_n and S_n are the spaces spanned by the functions $\cos \frac{k\pi}{T} x$, $k = 0, \ldots, n$ and $\sin \frac{k\pi}{T} x$, $k = 1, \ldots, n$ respectively. Since both $\cos \frac{k\pi}{T} x$ and $\sin \frac{(k+1)\pi}{T} x / \sin \frac{\pi}{T} x$ are polynomials of degree k in the variable $y = \cos \frac{\pi}{T} x$, it follows that the functions $h_{1,n}(y) = g_{e,n}(x)$ and $h_{2,n}(y) = g_{o,n}(x)/ \sin \frac{\pi}{T} x$ are polynomials of degree n in the variable $y \in [c(T), 1]$, where $c(T) = \cos \frac{\pi}{T}$. Moreover, since $g_{e,n}$ and $g_{o,n}$ are both defined by a least squares criterion, the change of variable $y = \cos \frac{\pi}{T} x$ gives that $h_{1,n}(y)$ and $h_{2,n}(y)$ are precisely the orthogonal projections of the functions $f_1(y) = f_e(x)$ and $f_2(y) = f_o(x)/ \sin \frac{\pi}{T} x$ respectively onto the space $\mathbb{P}_n$ of polynomials of degree n with respect to the weighted inner products

$$\langle g, h \rangle_1 = \int_{c(T)}^{1} g(x)h(x) \frac{1}{\sqrt{1 - y^2}} \,\mathrm{d}y,$$

$$\langle g, h \rangle_2 = \int_{c(T)}^{1} g(x)h(x) \sqrt{1 - y^2} \,\mathrm{d}y.$$

This argument gives an expression for the Fourier extension g_n in terms of polynomials in y. Note that, whilst the weight functions $w_1(y) = 1/\sqrt{1 - y^2}$ and $w_2(y) = \sqrt{1 - y^2}$ are identical to the Chebyshev weight functions of the first and second kinds, the domain $[c(T), 1]$ does not coincide with the standard unit interval $[-1, 1]$. In particular, the resulting orthogonal polynomial systems with respect to w_1 and w_2 are nonclassical. Yet, the similarity is striking, and, accordingly, the name *half-range Chebyshev* polynomials was introduced in [9].

There is at least one other similarity with Chebyshev polynomials. Recall that the Chebyshev approximation of a function $f(y)$ arises by applying the periodising transformation $y = \cos \theta$ and computing the Fourier series of the periodic function $f(\cos \theta)$. Here the situation is reversed. We approximate a function $f(x)$ by a Fourier series (i.e. the Fourier extension g_n), and, in order to analyse this approximation, relate it to a polynomial expansion in the variable $y = \cos \frac{\pi}{T} x$.

Let us return to the quantities $h_{i,n}$. Recall that the expansion of an analytic function g in (almost) any orthogonal polynomial system converges exponentially fast at rate ρ, where ρ is determined by the largest Bernstein ellipse (appropriately translated to $[c(T), 1]$) within which g is analytic. For the particular case of the functions $f_i(y)$,

$i = 1, 2$, the mapping $\cos^{-1} y$ introduces square-root type singularities at the point $y = -1$, which determine the value of this parameter. We have

Theorem 2.2 ([9], [1]) *Whenever the function f is sufficiently analytic, the approximation g_n defined by (2.1) satisfies*

$$|f(x) - g_n(x)| \sim E(T)^{-n}, \quad -1 \le x \le 1,$$

where $E(T) = \frac{3+c(T)+2\sqrt{2+2c(T)}}{1-c(T)}$.

This theorem confirms exponential convergence of the Fourier extension. As noted in [9], if f is not analytic in a sufficiently large Bernstein ellipse, the rate of exponential convergence is slower. This is of little concern to this paper, however, since we principally consider the entire function (1.1).

2.2 Numerical implementation

The most straightforward numerical implementation of the Fourier extension method involves solving the optimization problem (2.1) directly. This results in a $(2n+1) \times (2n+1)$ linear system for the coefficients of g_n, with corresponding matrix A.

The infinite system $\{\phi_j\}_{j=1}^{\infty}$, from which the Fourier extension g_n is computed, comprises a tight frame for the space $\mathrm{L}^2[-1,1]$ (with redundancy T). Hence, we expect the computation of g_n in this system to be severely ill-conditioned. In fact, numerical experiments indicate that the condition number $\kappa(A) \sim E(T)^{2n}$ for large n (note that $\dim G_n = 2n$) [9].

Whilst ill-conditioning cannot be avoided by a straightforward numerical method, it can be significantly ameliorated. The approach proposed in [1] is to define an extension $g_n \in G_n$ of f via the collocation conditions

$$g_n(x_i) = f(x_i), \quad i = 0, \ldots, 2n+2, \qquad (2.2)$$

where x_i are the so-called *symmetric mapped Chebyshev nodes*, given by

$$x_i = \frac{T}{\pi} \cos^{-1}\left[\frac{1}{2}(1-c(T))\cos\left(\frac{(2i+1)\pi}{2n+2}\right) + \frac{1}{2}(1+c(T))\right], \quad i = 0, \ldots, n,$$

and $x_{n+1+i} = -x_i$, $i = 0, \ldots, n$. Observe that, rather than computing polynomial expansions in the variable $y \in [c(T), 1]$ (as is the case for (2.1)), this approach computes polynomial interpolants in y at the set of Chebyshev nodes on $[c(T), 1]$. By standard results on polynomial approximation, the convergence rate of this approximation remains $E(T)^{-n}$. However, as shown in [1], the condition number of the corresponding linear system is significantly reduced: $\kappa(A) \sim E(T)^n$. In turn, this leads to significantly better numerical behaviour of approximation (2.2) over (2.1) [1].

We shall return to the issue of numerical computations in §4. For the moment, let us make one important observation. Since we approximate f in a frame, the inherent redundancy means that there will be infinitely many different representations of f. Thus, for large n, there will be many approximate representations of f from the set G_n. Hence, the exact Fourier extension defined by (2.1) may differ quite dramatically from that obtained via any particular numerical method.

3 Resolution power

Having introduced Fourier extensions, we now address resolution power and the value of the constant $r(T)$.

An immediate, albeit naïve, answer to this question is provided by Theorem 2.1. Since the function $f(x) = \exp(\mathrm{i}\pi\omega x)$ is smooth and satisfies $\|f\|_{\mathrm{H}^k[-1,1]} = \mathcal{O}\left((\pi\omega)^k\right)$, it follows that

$$\|f - g_n\|_{\mathrm{L}^2[-1,1]} \lesssim (\omega T n^{-1})^k, \quad k = 1, 2, \ldots.$$

The approximation g_n has $2n$ degrees of freedom, so we deduce that $r(T) \le 2T$. However, this estimate is only accurate for $T \approx 1$. In fact, for large T one witnesses $r(T) \approx \pi$. This comes as little surprise in view of the interpretation in terms of polynomials. As T increases, the spaces C_n and S_n both resemble polynomial spaces, and thus g_n behaves much like a polynomial expansion. The figure π arises directly as the well-known resolution constant of polynomial approximations [3, chpt. 2].

A better estimate for $r(T)$ can be derived by similar arguments to those used to establish exponential convergence of Fourier extension g_n. In particular, one obtains the following theorem:

Theorem 3.1 ([1]) *The resolution constant of the Fourier extension method (2.1) satisfies $r(T) \le T\sqrt{2-2c(T)}$. In particular, $r(T) = 2T + \mathcal{O}\left((T-1)^2\right)$ for $T \to 1$ and $r(T) = \pi + \mathcal{O}\left(T^{-2}\right)$ as $T \to \infty$.*

This theorem is verified in Figure 2.

4 Numerical examples

Theorem 3.1 concerns the resolution power of the exact Fourier extension (2.1), which, as discussed, may not

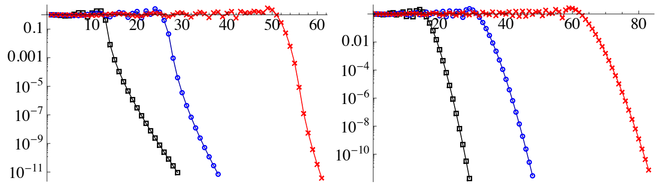

Figure 2: The quantity $\|f - g_n\|_\infty$ against $n = 1, 2, \ldots$, where f and g_n are given by (1.1) and (2.2) respectively, $T = \sqrt{2}$ (left), $T = 5$ (right) and $\omega = 10, 20, 40$.

coincide with the result of a given numerical implementation. This discrepancy has an important effect on resolution for larger T, as highlighted in Figure 3. Whilst $r(T)$ is approximately $2T$ for small T, when $T \gg 1$ the observed quantity is much larger than the value of π given by Theorem 3.1. In fact, when implemented this way, the Fourier extension method appears to satisfy $r(T) = 2T$ for all T, exactly as predicted by the naïve estimate in Theorem 2.1. This is confirmed in Figure 4.

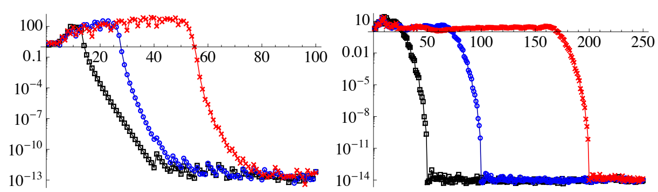

Figure 3: The quantity $\|f - g_n\|_\infty$ against n, where $T = \sqrt{2}$ (left), $T = 5$ (right) and $\omega = 10, 20, 40$.

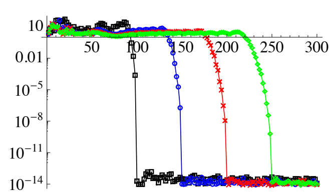

Figure 4: The quantity $\|f - g_n\|_\infty$ against n, where $f(x) = \exp(50\mathrm{i}\pi x)$ and $T = 2, 3, 4, 5$ (squares, circles, crosses and diamonds respectively).

To sum up, due to the inherent redundancy of the frame $\{\phi_j\}_{j=1}^{\infty}$, it appears challenging to obtain a resolution constant of π for large T with a straightforward numerical implementation. However, it is mainly the case of $T \approx 1$, which leads to the best resolution power, that is of interest in this paper. In this instance, as evidenced by Figures 2 and 3, the resolution constant is largely unaffected.

Recall that part of the motivation for this work was to improve the resolution constant of π for polynomial expansions. To show this improvement, in Figure 5 we compare the Fourier extension method to standard Chebyshev expansion of the oscillatory function

$$f(x) = (1 + x^2) \cos 10x \cos 100\pi x. \qquad (4.1)$$

As expected, the Fourier extension resolves f with fewer degrees of freedom. In particular, $n \approx 140$ gives ten digits of accuracy.

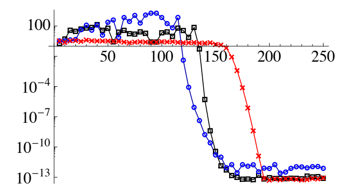

Figure 5: The quantity $\|f - g_n\|_\infty$, where g_n is the Chebyshev expansion (crosses), or the Fourier extension approximation with $T = \frac{4}{3}$ (squares), $T = \frac{8}{7}$ (circles).

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