Gibbs phenomenon and its removal for a class of orthogonal expansions

Ben Adcock
DAMTP, Centre for Mathematical Sciences
University of Cambridge
Wilberforce Rd, Cambridge CB3 0WA
United Kingdom

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Abstract

We detail the Gibbs phenomenon and its resolution for the family of orthogonal expansions consisting of eigenfunctions of univariate polyharmonic operators equipped with homogeneous Dirichlet boundary conditions. As we establish, it is possible to completely describe this phenomenon, including determining exact values for the size of the overshoot near both the domain boundary and the interior discontinuities of the function. Next, we demonstrate how the Gibbs phenomenon can be removed from such expansions using a number of different techniques. As a by-product, we introduce a generalisation of the classical Lidstone polynomials.

1 Introduction

Fourier series lie at the heart of countless methods in computational mathematics. Unfortunately, whenever a piecewise smooth function is represented by its Fourier series, the approximation suffers from the well-known Gibbs phenomenon [24, 36]. Several characteristics of this phenomenon include the slow convergence of the expansion away from the discontinuity locations, the lack of uniform convergence and the presence of $\mathcal{O}(1)$ oscillations near discontinuities [38]. In particular, the maximal overshoot of the Fourier series of a function f near any discontinuity x_0 is of size $c[f(x_0^+) - f(x_0^-)]$, where $c \approx 0.0895$.

It is a testament to the importance of the Gibbs phenomenon that the development of techniques for its amelioration, and indeed, complete removal, remains an active area of inquiry. The list of existing methods includes filtering [36], Gegenbauer reconstruction [20, 21], techniques based on extrapolation [14, 15, 16], Padé methods [13] and Fourier extension/continuation methods [10, 22], to name but a few (for a more comprehensive survey see [11, 36] and references therein). All such methods rely on one common principle: the Gibbs phenomenon is so regular, and so well understood mathematically, that it is possible to devise techniques to circumvent it.

As discussed in [20], the Gibbs phenomenon is certainly not restricted to Fourier series. Other notable examples include spherical harmonics, Fourier–Bessel series and radial basis functions [17]. In the same spirit, the intent of this paper is to study the Gibbs phenomenon in a certain family of orthogonal expansions. Specifically, we consider expansions of functions on $[-1,1]$ in eigenfunctions of univariate polyharmonic operators equipped with homogeneous Dirichlet boundary conditions:

$$(-1)^q \phi^{(2q)}(x) = \mu\phi(x), \quad x \in [-1,1], \quad \phi^{(r)}(\pm 1) = 0, \quad r = 0,\ldots,q-1, \quad q \in \mathbb{N}_+. \tag{1.1}$$

Denoting the n^{th} such eigenfunction by ϕ_n, the collection of sets $\{\phi_n\}_{n=1}^{\infty}$ forms a one-parameter family of orthogonal bases, with parameter q.

Such eigenfunctions (and the corresponding expansions) have recently been developed in detail in [5, 6]. In [6] explicit forms for the eigenfunctions were given, and methods for computing expansion coefficients provided (based on combinations of classical and highly oscillatory quadratures). Potential applications to the numerical solution of differential and integral equations were also discussed. Additionally, a number of approximation-theoretic properties of such expansions were established in [5].

As we demonstrate in this paper, the Gibbs phenomenon occurs in such expansions in a similar, but not identical, manner to the case of Fourier series. In particular, interior discontinuities of the function lead to a virtually identical phenomenon. Conversely, near the endpoints, the phenomenon has a number of important distinctions. Nonetheless, in both cases we are able to exactly determine the overshoot constant c.

When $q = 1$, (1.1) corresponds to eigenfunctions of the Laplace operator subject to homogeneous Dirichlet boundary conditions. This results in the basis $\{\cos(n - \frac{1}{2})\pi x : n \in \mathbb{N}_+\} \cup \{\sin n\pi x : n \in \mathbb{N}_+\}$. This basis has been employed in [2, 4] in the numerical solution of second order boundary value problems. In [34] it was applied to the solution of interior Helmholtz problems in polygonal domains. The close proximity of this basis to the standard Fourier basis $\{\cos n\pi x : n \in \mathbb{N}\} \cup \{\sin n\pi x : n \in \mathbb{N}_+\}$ highlights that a similar Gibbs phenomenon ought to occur in expansions in so-called *Laplace–Dirichlet* eigenfunctions. However, as we prove, even for arbitrary $q \geq 1$ (when the eigenfunctions (1.1) are no longer simple trigonometric functions), the similarity to the classical Gibbs phenomenon persists.

Unfortunately, even when the approximated function f is smooth on $[-1, 1]$, the Gibbs phenomenon still occurs at the domain endpoints $x = \pm 1$ (had this comment been in reference to Fourier series, we would have interpreted f as a function defined on the unit torus $\mathbb{T} = [-1, 1)$ with jump discontinuity at $x = -1$). For this reason, the second part of this paper is devoted to the removal of the Gibbs phenomenon from such expansions. As discussed in [6, 23], a simple remedy exists for its amelioration. We merely replace the so-called *polyharmonic–Dirichlet* eigenfunctions (1.1) with those corresponding to homogeneous Neumann boundary conditions

$$(-1)^q \phi^{(2q)}(x) = \mu \phi(x), \quad x \in [-1, 1], \quad \phi^{(q+r)}(\pm 1) = 0, \quad r = 0, \ldots, q-1, \quad q \in \mathbb{N}_+. \tag{1.2}$$

As shown in [2, 5], the expansion of a smooth function f in polyharmonic–Neumann eigenfunctions converges uniformly on $[-1, 1]$, as do its first $q - 1$ derivatives. However, whilst the Gibbs phenomenon is no longer present in the expansion itself, it occurs in the q^{th} derivative. As it turns out, this derivative corresponds precisely to the expansion of the $f^{(q)}$ in the polyharmonic–Dirichlet eigenfunctions (1.1). Thus, the Gibbs phenomenon is only mitigated by this procedure, not removed completely. Nonetheless, in some applications, this approach has been found to be beneficial. So-called *modified Fourier expansions* (expansions in the eigenfunctions (1.2) corresponding to $q = 1$) have been found to confer a number of advantages over more standard schemes when applied to the numerical solution of differential equations [2] and integral equations [12].

Despite the improvement offered by polyharmonic–Neumann eigenfunctions (1.2), the second part of this paper is devoted to the complete removal of the Gibbs phenomenon from polyharmonic–Dirichlet expansions. As we discuss, factors such as smoothness and periodicity that determine the convergence rate of Fourier series have natural analogues for these expansions. Once such factors are understood, it is possible to generalise a number of known techniques to the polyharmonic–Dirichlet case. In doing so, we highlight the broad applicability of such techniques, beyond their original purpose. The culmination of this work is a spectrally accurate approximation scheme based on such eigenfunctions, similar in character to the Fourier extension method [10, 22]. Moreover, as a by-product, families of polynomials that generalise the classical Lidstone polynomials are introduced and discussed.

The outline of the remainder of this paper is as follows. In Section 2 we introduce expansions in polyharmonic–Dirichlet eigenfunctions and recap salient aspects of [5, 6]. Pointwise convergence of such expansions away from discontinuities is established in Section 3, and in Section 4 we detail the Gibbs phenomenon. Techniques to resolve the Gibbs phenomenon are studied in Section 5.

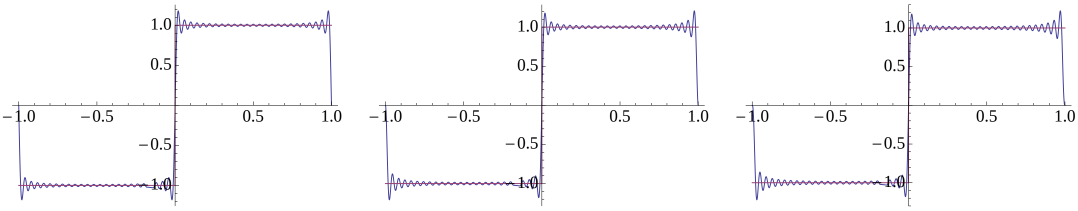

Figure 1: The approximation $f_{50}(x)$ against $x \in [-1, 1]$ for $q = 1, 2, 3$ (left to right), where f is the Heavyside function.

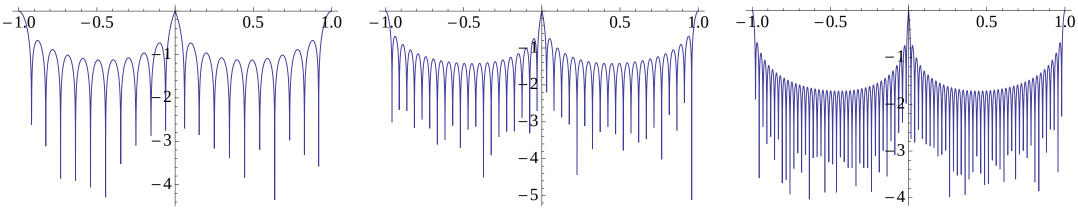

Figure 2: The error $\log_{10} |f(x) - f_N(x)|$ for $N = 10, 20, 40$ (left to right), where $q = 2$ and f is the Heavyside function.

2 Polyharmonic–Dirichlet expansions

Polyharmonic eigenfunctions have been studied systematically in [6]. Since the polyharmonic operator, when equipped with homogeneous Dirichlet boundary conditions, is positive definite, it has a countable number of positive eigenvalues μ_n (having no finite limit point in $\mathbb{R}$) with corresponding eigenfunctions ϕ_n forming an orthogonal basis of $\mathrm{L}^2(-1, 1)$. For this reason, any function $f \in \mathrm{L}^2(-1, 1)$ may be expanded in polyharmonic–Dirichlet eigenfunctions

$$f(\cdot) \sim \sum_{n=1}^{\infty} \frac{\hat{f}_n}{\|\phi_n\|^2} \phi_n(\cdot),$$

with identification in the usual L^2 sense. Here $\hat{f}_n = \int_{-1}^{1} f(x)\overline{\phi_n(x)}\,\mathrm{d}x$ is the coefficient of f with respect to ϕ_n, $\|g\|^2 = \int_{-1}^{1} |g(x)|^2\,\mathrm{d}x$ is the standard L^2 norm of $g \in \mathrm{L}^2(-1, 1)$ and $\bar{z}$ denotes the complex conjugate of $z \in \mathbb{C}$. In practice, this infinite sum must be truncated, leading to

$$f_N(x) = \sum_{n=1}^{N} \frac{\hat{f}_n}{\|\phi_n\|^2} \phi_n(x). \tag{2.1}$$

Our interest in the first part of this paper lies with the convergence of the approximation f_N to f (or lack thereof) and, in particular, the nature of the Gibbs phenomenon. To this end, in Figures 1 and 2 we highlight the approximation of the Heavyside function using polyharmonic–Dirichlet eigenfunctions. As is evident, $\mathcal{O}(1)$ oscillations occur near both the endpoints $x = \pm 1$ and the interior discontinuity $x = 0$. Such oscillations are indicative of the Gibbs phenomenon, as shall be described in greater detail in Section 4. Conversely, as we prove in Section 3, pointwise convergence occurs away from $x = -1, 0, 1$.

Before doing so, however, we first require explicit expressions for the polyharmonic–Dirichlet eigenfunctions, valid for arbitrary q. Trivially, we may write the n^{th} eigenfunction ϕ_n as

$$\phi_n(x) = \sum_{r=0}^{2q-1} c_{r,n} \mathrm{e}^{\lambda_r \alpha_n x}, \tag{2.2}$$

where $\lambda_r = -\mathrm{i}\mathrm{e}^{\frac{\mathrm{i}\pi}{q}}$, and the constants $\alpha_n^{2q} = \mu_n$ and the $c_{r,n} \in \mathbb{C}$ are specified by enforcing boundary conditions. This results in an algebraic eigenproblem (for each n), from which such coefficients can be computed. As discussed in [6], this expression is usually reduced to a real form for computations. However, for the purposes of analysis, it is significantly simpler to retain the complex exponential version (2.2).

In [5] the asymptotic nature (as $n \to \infty$) of the eigenfunctions ϕ_n and the values α_n were considered. It was found that such quantities have known asymptotic expressions up to exponentially small terms in n. In particular, if $\gamma_q = \sin\frac{\pi}{q}$, then

$$\alpha_n = \frac{1}{4}(2n+q-1)\pi + \mathcal{O}\left(\mathrm{e}^{-n\pi\gamma_q}\right), \quad n \to \infty, \tag{2.3}$$

and

$$\phi_n(x) = \sum_{r=0}^{q-1} c_r \left[\mathrm{e}^{\lambda_r\alpha_n(x-1)} + (-1)^{n+1}\mathrm{e}^{-\lambda_r\alpha_n(x+1)}\right] + \mathcal{O}\left(\mathrm{e}^{-n\pi\gamma_q}\right), \tag{2.4}$$

uniformly for $x \in [-1,1]$. Here the values c_r are given explicitly as particular minors of the matrix $V \in \mathbb{C}^{q\times q}$ with $(r,s)^{\text{th}}$ entry λ_r^s. In fact, $c_r = (-1)^q(\det V)(V^{-1})_{r,q}$. Several other results concerning such eigenfunctions were also obtained in [5]. In particular, away from the endpoints $x = \pm 1$, the n^{th} eigenfunction ϕ_n is exponentially close to a regular oscillator:

$$\phi_n(x) = c_0\left[\mathrm{e}^{-\mathrm{i}\alpha_n(x-1)} + (-1)^{n+1}\mathrm{e}^{\mathrm{i}\alpha_n(x+1)}\right] + \mathcal{O}\left(\mathrm{e}^{-\frac{1}{2}n\pi\gamma_q(1-|x|)}\right).$$

Furthermore,

$$\phi_n^{(r)}(x) = c_0(-\mathrm{i})^r\alpha_n^r\left[\mathrm{e}^{-\mathrm{i}\alpha_n(x-1)} + (-1)^{n+r+1}\mathrm{e}^{\mathrm{i}\alpha_n(x+1)}\right] + \mathcal{O}\left(n^r\mathrm{e}^{-\frac{1}{2}n\pi\gamma_q(1-|x|)}\right), \quad r \in \mathbb{N}. \tag{2.5}$$

and, at the endpoints $x = \pm 1$, the function ϕ_n and its derivatives are given explicitly by

$$\phi_n^{(r)}(\pm 1) = (\pm 1)^{n+r} d_r\alpha_n^r, \quad d_r = \mathrm{i}^{q+r+1}c_0 + \sum_{s=0}^{q-1} c_s\lambda_s^r, \quad r \in \mathbb{N}. \tag{2.6}$$

Finally, concerning the eigenfunction norm, we have $\|\phi_n\| = c + \mathcal{O}(\mathrm{e}^{-\frac{1}{2}n\pi\gamma_q})$, where $c = 2|c_0|$ [5].

We remark at this point that such exponential asymptotics are vital to this paper. We are able to detail the Gibbs phenomenon for the family of polyharmonic–Dirichlet expansions precisely because such remainder terms decay so rapidly with n (for convenience, from this point onwards we drop any exponentially small terms). Having said this, whilst the classical Gibbs phenomenon for Fourier series is usually studied by analysing finite sums with indices $n = 1, \ldots, N$, we need to consider infinite sums with $n > N$ so as to exploit (2.3)–(2.6). Although additional care is necessary to ensure convergence, few technical issues arise from this approach.

3 Pointwise convergence

The first facet of the classical Gibbs phenomenon is the lack of uniform convergence on $[-1,1]$ of truncated Fourier sums. Moreover, whilst such expansions converge pointwise away from the discontinuities, the rate of convergence is only linear in the truncation parameter N.

The intent of this section is to demonstrate identical convergence of polyharmonic–Dirichlet expansions. To this end, suppose that $f : [-1,1] \to \mathbb{R}$ is piecewise analytic with jump discontinuities at $-1 < x_1 < \ldots x_k < 1$ (we could impose lower regularity in each subdomain, yet, for simplicity, we shall assume analyticity throughout). In addition, let $x_0 = -1$ and $x_{k+1} = 1$. To study the expansion f_N of f, it is necessary to obtain explicit expressions for the coefficients $\hat{f}_n$. These are provided by first replacing ϕ_n by $(-1)^q\alpha_n^{-2q}\phi_n^{(2q)}$ in $\int_{-1}^1 f(x)\phi_n(x)\,\mathrm{d}x$ and integrating by parts. Taking care of the discontinuities, this gives

$$\begin{aligned}\int_{-1}^1 f(x)\phi_n(x)\,\mathrm{d}x =& \frac{(-1)^q}{\alpha_n^{2q}}\Bigg\{f(1)\phi_n^{(2q-1)}(1) - f(-1)\phi_n^{(2q-1)}(-1) + \\ &-\sum_{j=1}^k [f](x_j)\phi_n^{(2q-1)}(x_j)\Bigg\} + \frac{(-1)^{q+1}}{\alpha_n^{2q}}\sum_{j=0}^k\int_{x_j}^{x_{j+1}} f'(x)\phi_n^{(2q-1)}(x)\,\mathrm{d}x,\end{aligned} \tag{3.1}$$

where $[g](x) = g(x^+) - g(x^-)$. Integrating by parts a further $2q-1$ times and applying the boundary conditions $\phi_n^{(s)}(\pm 1) = 0$, $s = 0, \ldots, q-1$, we then find that

$$\begin{aligned}\int_{-1}^{1} f(x)\phi_n(x)\,\mathrm{d}x =& \frac{(-1)^q}{\alpha_n^{2q}} \Bigg\{ \sum_{s=0}^{q-1} (-1)^s \left[f^{(s)}(1)\phi_n^{(2q-s-1)}(1) - f^{(s)}(-1)\phi_n^{(2q-s-1)}(-1) \right] \\ &- \sum_{s=0}^{2q-1} (-1)^s \left[f^{(s)} \right] (x_j)\phi_n^{(2q-s-1)}(x_j) \Bigg\} + \frac{(-1)^q}{\alpha_n^{2q}} \sum_{j=0}^{k} \int_{x_j}^{x_{j+1}} f^{(2q)}(x)\phi_n(x)\,\mathrm{d}x.\end{aligned}$$

Furthermore, after iterating this process, we obtain

$$\begin{aligned}&\int_{-1}^{1} f(x)\phi_n(x)\,\mathrm{d}x \\ &\sim \sum_{r=0}^{\infty} \Bigg\{ \sum_{s=0}^{q-1} \frac{(-1)^{(r+1)q+s}}{\alpha_n^{2(r+1)q}} \left[f^{(2rq+s)}(1)\phi_n^{(2q-s-1)}(1) - f^{(2rq+s)}(-1)\phi_n^{(2q-s-1)}(-1) \right] \\ &\quad - \sum_{s=0}^{2q-1} \frac{(-1)^{(r+1)q+s}}{\alpha_n^{2(r+1)q}} \sum_{j=1}^{k} \left[f^{(2rq+s)} \right] (x_j)\phi_n^{(2q-s-1)}(x_j) \Bigg\}, \quad n \to \infty. \end{aligned} \tag{3.2}$$

In [5], it was shown that $\|\phi_n^{(r)}\|_\infty = \mathcal{O}(n^r)$. Since $\alpha_n = \mathcal{O}(n)$, (3.2) presents an asymptotic expansion for the coefficient $\hat{f}_n$ in inverse powers of n (in the usual Poincaré sense). Hence, we use the symbol $\sim$. Naturally, the right hand side of (3.2) will not typically converge for fixed n.

On closer inspection, (3.2) also provides several indications as to the nature of the Gibbs phenomenon for polyharmonic–Dirichlet expansions. First, the endpoints $x = \pm 1$ are represented in a fundamentally different manner to the internal discontinuities x_j, $j = 1, \ldots, k$. In particular, whilst the contribution from interior discontinuities involves only the jump values $[f^{(2rq+s)}](x_j)$, the values $f^{(2rq+s)}(\pm 1)$ occur separately. We may therefore expect, and it turns out to the the case, that the maximum overshoot size at each endpoint depends only on the value of f at that endpoint. Conversely, in a manner akin to Fourier series, the overshoot at any internal discontinuity x_j involves the jump value $[f](x_j)$.

This last point comes as no great surprise. As shown in (2.5), the eigenfunctions ϕ_n are (up to exponentially small terms) regular oscillators in $(-1, 1)$. Hence, intuition suggests that the error $f(x) - f_N(x)$ will behave similarly to a tail of a standard Fourier series (this is a phenomenon known as *equiconvergence* [28, 37]). In contrast, (2.5) breaks down near the endpoints, and a different treatment is necessary. Nonetheless, we are still able to detail the Gibbs phenomenon in this case, as we consider in Section 4.2.

Returning to pointwise convergence, suppose that $U_j \subseteq [-1, 1]$, $j = 0, \ldots, k+1$, is compact and $U_j \cap \{x_j\} = \emptyset$. Given $N \in \mathbb{N}_+$, we now define

$$\Theta_N(j, r, s; x) = \frac{1}{c^2} \sum_{n \geq N} \frac{\overline{\phi_n^{(2q-s-1)}(x_j)}}{\alpha_n^{2(r+1)q}} \phi_n(x), \quad r \in \mathbb{N}, \quad s = 0, \ldots, 2q-1. \tag{3.3}$$

It is not immediately apparent that such functions are well-defined. However

Lemma 1. *For each $j = 0, \ldots, k+1$, $r \in \mathbb{N}$ and $s = 0, \ldots, 2q-1$, the function $\Theta_N(j, r, s; \cdot)$ is well-defined and continuous on U_j. Moreover,*

$$\Theta_N(j, r, s; x) = \mathcal{O}\left(N^{-2rq-s-1}\right), \tag{3.4}$$

uniformly for $x \in U_j$.

To prove this lemma, it is first useful to note the following:

Lemma 2. *Suppose that V is a compact subset of $\{z \in \mathbb{C} : |z| \leq 1, z \neq 1\}$. Then the function*

$$F_{r,a}(z) = \sum_{n=0}^{\infty} \frac{z^n}{(n+a)^r}, \quad a > 0, \quad r \geq 1$$

is well-defined and continuous on V. Moreover, if $\sigma_{z,r}(t) = \frac{t^{r-1}}{1-z\mathrm{e}^{-t}}$, then

$$F_{r,a}(z) \sim \frac{1}{\Gamma(r)} \sum_{s=r-1}^{\infty} \frac{\sigma_{z,r}^{(r-1)}(0)}{a^{s+1}}, \quad a \to \infty.$$

In particular, $F_a(z) = a^{-r}(1-z)^{-1} + \mathcal{O}\left(a^{-2}\right)$.

This lemma is very similar to a result proved in [30]. The only generalisations are allowing $|z| \leq 1$, as opposed to $|z| = 1$, and $r \geq 1$ instead of $r > 1$. The proof is virtually identical, hence is omitted. Note that $F_{r,a}(z)$ is precisely the Lerch transcendental function $\Phi(z, r, a)$ [35].

Proof of Lemma 1. Consider first the case $j = 1, \ldots, k$. By (2.5), it follows that

$$\overline{\phi_n^{(2q-s-1)}(x_j)} = c_0 \mathrm{i}^{2q-s-1} \alpha_n^{2q-s-1} \left[\mathrm{e}^{\mathrm{i}\alpha_n(x_j-1)} + (-1)^{n+s} \mathrm{e}^{-\mathrm{i}\alpha_n(x_j+1)}\right].$$

Now consider the partial sums

$$\sum_{n=N}^{N+M} \alpha_n^{-2rq-s-1} \left[\mathrm{e}^{\mathrm{i}\alpha_n(x_j-1)} + (-1)^{n+s} \mathrm{e}^{-\mathrm{i}\alpha_n(x_j+1)}\right] \phi_n(x).$$

Replacing ϕ_n by its asymptotic expression (2.4), we obtain

$$\sum_{l=0}^{q-1} \sum_{n=N}^{N+M} \alpha_n^{-2rq-s-1} \left[\mathrm{e}^{\mathrm{i}\alpha_n(x_j-1)} + (-1)^{n+s} \mathrm{e}^{-\mathrm{i}\alpha_n(x_j+1)}\right] \left[\mathrm{e}^{\lambda_l \alpha_n(x-1)} + (-1)^{n+1} \mathrm{e}^{-\lambda_l \alpha_n(x+1)}\right],$$

up to exponentially small terms in N. Thus, it suffices to separately consider the four sums

$$\sum_{n=N}^{N+M} \alpha_n^{-2rq-s-1} \mathrm{e}^{[\mathrm{i}(x_j-1)+\lambda_l(x-1)]\alpha_n}, \quad \sum_{n=N}^{N+M} \alpha_n^{-2rq-s-1} (-1)^n \mathrm{e}^{[-\mathrm{i}(x_j+1)+\lambda_l(x-1)]\alpha_n}$$

$$\sum_{n=N}^{N+M} \alpha_n^{-2rq-s-1} (-1)^n \mathrm{e}^{[\mathrm{i}(x_j-1)-\lambda_l(x+1)]\alpha_n}, \quad \sum_{n=N}^{N+M} \alpha_n^{-2rq-s-1} \mathrm{e}^{[-\mathrm{i}(x_j+1)-\lambda_l(x+1)]\alpha_n}.$$

Since all cases are similar, we consider the first sum only. As $\alpha_n \sim \frac{1}{4}(2n+q-1)\pi$, we see that this reduces to a constant multiple of

$$z^{(M+\frac{q-1}{2})\pi} \sum_{m=0}^{M} \frac{z^m}{(m+M+\frac{q-1}{2})^{2kq+r+1}},$$

where $z = \mathrm{e}^{\frac{1}{2}[\lambda_l(x-1)+\mathrm{i}(x_j-1)]\pi}$. Now, since $\mathrm{Re}\,\lambda_l \geq 0$ and $x \leq 1$, we conclude that $|z| \leq 1$. Moreover, if $l = 1, \ldots, q-1$, then $\mathrm{Re}\,\lambda_l > 0$. Hence, $|z| < 1$ in this case as $x \neq 1$. Now suppose that $l = 0$, so that $z = \mathrm{e}^{\frac{1}{2}\mathrm{i}(x_j-x)\pi}$. Since $x \neq x_j$, it follows that $z \neq 1$. Thus, an application of Lemma 2 now confirms that this sum converges uniformly on U_j to a continuous function. In addition, we also obtain the asymptotic estimate (3.4).

It remains to demonstrate the result when $j = 0, k+1$. Both cases are similar, so we consider $j = k+1$, whence $x_j = 1$. Consider the partial sum

$$\sum_{n=N}^{N+M} \frac{\overline{\phi_n^{(2q-s-1)}(1)}}{\alpha_n^{2(r+1)q}} \phi_n(x) = \overline{d_{2q-s-1}} \sum_{l=0}^{q-1} \sum_{n=N}^{N+M} \alpha_n^{-2rq-s-1} \left[\mathrm{e}^{\lambda_l \alpha_n(x-1)} + (-1)^n \mathrm{e}^{-\lambda_l \alpha_n(x+1)}\right].$$

We now proceed in an identical manner. □

With this lemma to hand, we are now able to provide the key result of this section:

Theorem 1. *Suppose that $U \subseteq [-1, 1]$ is compact and $\{x_0, \ldots, x_{k+1}\} \cap U = \emptyset$. Then f_N converges uniformly to f in U. In particular, $f(x) - f_N(x) = \mathcal{O}\left(N^{-1}\right)$ uniformly for $x \in U$.*

Proof. Recall (3.1). Integrating the remainder term by parts once more, we have

$$\hat{f}_n = \frac{(-1)^q}{\alpha_n^{2q}} \left\{ f(1)\overline{\phi_n^{(2q-1)}(1)} - f(-1)\overline{\phi_n^{(2q-1)}(-1)} - \sum_{j=1}^{k} [f](x_j)\overline{\phi_n^{(2q-1)}(x_j)} \right\} + \mathcal{O}\left(n^{-2}\right).$$

Substituting this into (2.1), we find that

$$\begin{aligned} f_{N+M}(x) - f_N(x) = (-1)^q \Bigg\{ & f(1)\left[\Theta_{N+M}(k+1,0,0;x) - \Theta_N(k+1,0,0;x)\right] \\ & - f(-1)\left[\Theta_{N+M}(0,0,0;x) - \Theta_N(0,0,0;x)\right] \\ & - \sum_{j=1}^{k} [f](x_j)\left[\Theta_{N+M}(j,0,0;x) - \Theta_N(j,0,0;x)\right] \Bigg\} + \mathcal{O}\left(N^{-1}\right). \end{aligned}$$

Using Lemma 1, we deduce that $\{f_N(x)\}_{N=1}^{\infty}$ forms a Cauchy sequence. Hence f_N converges uniformly on U to some continuous function $\tilde{f}$. Now suppose that $\tilde{f}(y) \neq f(y)$ for some $y \in U$. Then, by continuity, these functions must differ on some neighbourhood $U' \subseteq U$. Hence

$$0 < \int_{U'} |f(x) - \tilde{f}(x)|^2 \, \mathrm{d}x \leq \lim_{N\to\infty} \int_{-1}^{1} |f(x) - f_N(x)|^2 \, \mathrm{d}x = 0,$$

a contradiction (the rightmost equality follows from the fact that $\{\phi_n\}$ is an orthogonal basis of $\mathrm{L}^2(-1,1)$ and f_N is the orthogonal projection). Hence $\tilde{f} = f$ and we conclude uniform convergence of f_N to f in U.

Moreover, we may now write the error $f(x) - f_N(x)$ as an infinite sum. In an identical manner, we find that

$$\begin{aligned} f(x) - f_N(x) = (-1)^q \Bigg\{ & f(1)\Theta_N(k+1,0,0;x) - f(-1)\Theta_N(0,0,0;x) \\ & - \sum_{j=1}^{k} [f](x_j)\Theta_N(j,0,0;x) \Bigg\} + \mathcal{O}\left(N^{-1}\right). \end{aligned} \tag{3.5}$$

In view of Lemma 1, we conclude that the rate of convergence is $\mathcal{O}\left(N^{-1}\right)$. □

Though pointwise convergence has now been confirmed, we can actually provide a far more detailed assessment. Trivially, using (3.2), we find that

$$\begin{aligned} & f(x) - f_N(x) \\ & \sim \sum_{r=0}^{\infty} \Bigg\{ \sum_{s=0}^{q-1} (-1)^{(r+1)q+s} \left\{ f^{(2rq+s)}(1)\Theta_N(k+1,r,s;x) - f^{(2rq+s)}(-1)\Theta_N(0,r,s;x) \right\} \\ & \qquad - \sum_{s=0}^{2q-1} (-1)^{(r+1)q+s} \sum_{j=1}^{k} \left[f^{(2rq+s)} \right](x_j)\Theta_N(j,r,s;x) \Bigg\}, \quad N \to \infty. \end{aligned} \tag{3.6}$$

Due to Lemma 1, this is an asymptotic expansion for $f(x) - f_N(x)$, valid uniformly in U.

With sufficient effort, we could derive exact expressions for each Θ_N in terms of the Lerch transcendental function $\Phi(\cdot,\cdot,\cdot)$. However, we shall not do this (this is described in further detail in [5, 30]). Nonetheless, it is of interest to determine the precise leading order asymptotic behaviour of such functions. In turn, this provides an exact expression for the leading order behaviour of the error $f(x) - f_N(x)$. Recalling (3.5), we note that this behaviour is determined solely by the functions $\Theta_N(j,0,0;x)$, $j = 0,\ldots,k+1$. The only contribution of f occurs in the values $f(\pm 1)$ and $[f](x_j)$, $j = 1,\ldots,k$. Hence, we conclude that, aside function dependent constants, the local behaviour of the error independent of the approximated function (naturally, it is also dependent on the singularity locations $x_1,\ldots,x_k$). For this reason, it suffices to consider only the functions $\Theta_N(j,0,0;x)$, $j = 0,\ldots,k+1$. We have

Lemma 3. *For $j=1,\dots,k$ the function $\Theta_N(j,0,0;x)$, $x\in U_j$, satisfies*

$$\Theta_N(j,0,0;x)$$
$$=\frac{1}{4}(-1)^{q+1}c^{-2}\left\{\frac{\cos(\alpha_N-\frac{\pi}{4})(x-x_j)}{\sin\frac{\pi}{4}(x-x_j)}+(-1)^N\frac{\sin(\alpha_N-\frac{\pi}{4})(x+x_j)}{\cos\frac{\pi}{4}(x+x_j)}\right\}\alpha_N^{-1}+\mathcal{O}\left(N^{-2}\right).$$

Moreover,

$$\Theta_N(k+1,0,0;x)=-\mathrm{i}\overline{d_{2q-1}}c_0c^{-2}\frac{\cos\left[\alpha_N x-\frac{\pi}{4}(x+1)\right]}{\cos\frac{\pi}{4}(x+1)}\mathrm{e}^{\mathrm{i}\alpha_N}\alpha_N^{-1}+\mathcal{O}\left(N^{-2}\right),$$
$$\Theta_N(0,0,0;x)=-\mathrm{i}\overline{d_{2q-1}}c_0c^{-2}\frac{\sin\left[\alpha_N x-\frac{\pi}{4}(x+1)\right]}{\sin\frac{\pi}{4}(x+1)}(-1)^N\mathrm{e}^{\mathrm{i}\alpha_N}\alpha_N^{-1}+\mathcal{O}\left(N^{-2}\right),$$

for $x\in U_{k+1}$ and $x\in U_0$ respectively.

Proof. Suppose that $w\in\mathbb{C}$ with $|w|\le 1$ and $w\neq 1$. Consider $\sum_{n\ge N}(\pm1)^n w^{\alpha_n}\alpha_n^{-1}$. Since $\alpha_{n+N}=\alpha_N+\frac{1}{2}n\pi$, we find that

$$\sum_{n\ge N}\frac{(\pm1)^n w^{\alpha_n}}{\alpha_n}=\frac{2(\pm1)^N w^{\alpha_N}}{\pi}\sum_{n=0}^{\infty}\frac{(\pm w^{\frac{1}{2}\pi})^m}{n+N+\frac{q-1}{2}}=\frac{2(\pm1)^N w^{\alpha_N}}{\pi}F_{1,N+\frac{q-1}{2}}\left(\pm w^{\frac{1}{2}\pi}\right).$$

It now follows from Lemma 2 that

$$\sum_{n\ge N}\frac{(\pm1)^n w^{\alpha_n}}{\alpha_n}=\frac{(\pm1)^N w^{\alpha_N}}{\alpha_N(1\mp w^{\frac{1}{2}\pi})}+\mathcal{O}\left(N^{-2}\right). \tag{3.7}$$

The full result is obtained upon substituting the asymptotic formulae (2.4)–(2.6) for ϕ_n into (3.3) and applying (3.7) to each term. To simplify the various expressions, we use the equalities

$$\frac{\mathrm{e}^{\mathrm{i}a}}{1+\mathrm{e}^{\mathrm{i}b}}+\frac{\mathrm{e}^{-\mathrm{i}a}}{1+\mathrm{e}^{\mathrm{i}b}}=\frac{\cos(a-\frac{1}{2}b)}{\cos\frac{1}{2}b},\quad \frac{\mathrm{e}^{\mathrm{i}a}}{1+\mathrm{e}^{\mathrm{i}b}}-\frac{\mathrm{e}^{-\mathrm{i}a}}{1+\mathrm{e}^{\mathrm{i}b}}=\mathrm{i}\frac{\sin(a-\frac{1}{2}b)}{\cos\frac{1}{2}b},$$
$$\frac{\mathrm{e}^{\mathrm{i}a}}{1-\mathrm{e}^{\mathrm{i}b}}+\frac{\mathrm{e}^{-\mathrm{i}a}}{1-\mathrm{e}^{\mathrm{i}b}}=-\frac{\sin(a-\frac{1}{2}b)}{\sin\frac{1}{2}b},\quad \frac{\mathrm{e}^{\mathrm{i}a}}{1-\mathrm{e}^{\mathrm{i}b}}-\frac{\mathrm{e}^{-\mathrm{i}a}}{1-\mathrm{e}^{\mathrm{i}b}}=\mathrm{i}\frac{\cos(a-\frac{1}{2}b)}{\sin\frac{1}{2}b},$$

which are valid for all $a,b\in\mathbb{C}$, $b\neq(2n+1)\pi$ (top) and $b\neq 2n\pi$ (bottom), $n\in\mathbb{Z}$. □

In Figure 3 we display the pointwise error in approximating the function

$$f(x)=\begin{cases}-1 & -1\le x<-\frac{1}{4}\\ x^2 & \frac{1}{4}\le x<\frac{1}{3}\\ -\mathrm{e}^{-\frac{1}{2}x} & \frac{1}{3}\le x\le 1\end{cases} \tag{3.8}$$

with polyharmonic–Dirichlet eigenfunctions corresponding to $q=1,2,3$. Several key features of polyharmonic–Dirichlet expansions are now apparent. First, away from the discontinuities (and $x=\pm1$) the error is qualitatively similar, regardless of q. Moreover, the error oscillates with $\mathcal{O}(N)$ frequency, and the particular bounding curve increases as x approaches x_j, $j=0,\dots,k+1$. All these features are predicted by the previous lemma, with latter two being consequences of the terms $\mathrm{e}^{\pm\mathrm{i}\alpha_N x}$, which account for the oscillations, and the denominators $\sin\frac{\pi}{4}(x-x_j)$,$\cos\frac{\pi}{4}(x+x_j)$, $\cos\frac{\pi}{4}(x+1)$ and $\sin\frac{\pi}{4}(x+1)$, which are unbounded as $x\to x_j$, respectively.

4 The Gibbs phenomenon

We now study the Gibbs phenomenon in polyharmonic–Dirichlet expansions. The main facet of this is determining the maximal overshoot of the expansion f_N near the points $x_0,\dots,x_{k+1}$. As previously indicated, this phenomenon is fundamentally different at the interior discontinuities $x_1,\dots,x_k$ than at the endpoints $x=\pm1$. In particular, as we now demonstrate, the interior Gibbs phenomenon is virtually identical to that occurring in classical Fourier series.

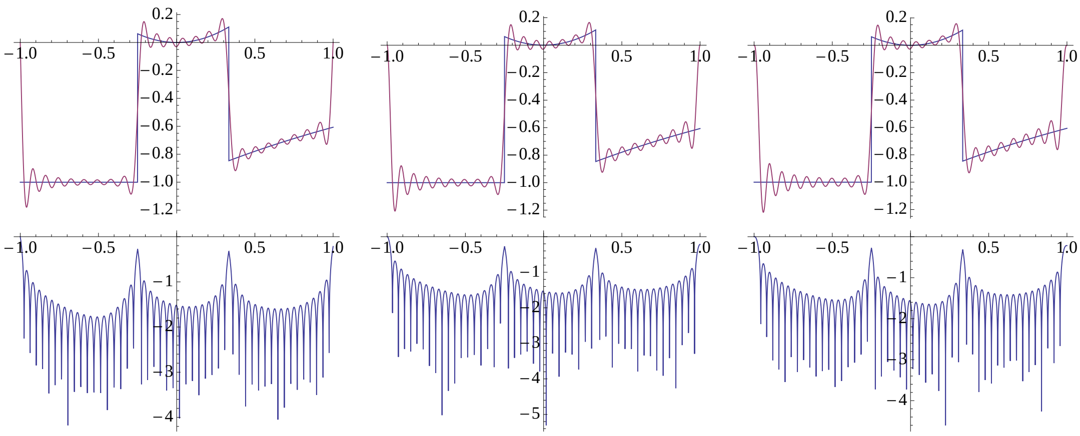

Figure 3: Top row: the functions $f(x)$ and $f_{25}(x)$ for $q = 1, 2, 3$ (left to right), where $f(x)$ is given by (3.8). Bottom row: the error $|f(x) - f_{25}(x)|$ against $x \in [-1, 1]$.

4.1 Interior Gibbs phenomenon

Suppose that $x \in U \backslash \{x_j\}$, where U is a compact neighbourhood of x_j (for some $j = 1, \ldots, k$) with $x_l \notin U$ for $l \neq j$, $l = 0, \ldots, k + 1$. Using (3.5) and Lemma 1, we find that

$$f(x) - f_N(x) = \frac{(-1)^{q+1}[f](x_j)}{c^2} \sum_{n>N} \frac{\overline{\phi_n^{(2q-1)}(x_j)}}{\alpha_n^{2q}} \phi_n(x) + \mathcal{O}\left(N^{-1}\right). \tag{4.1}$$

By (2.5) it follows that

$$\begin{aligned} \overline{\phi_n^{(2q-1)}(x_j)} \phi_n(x) = |c_0|^2 \mathrm{i}^{2q-1} \alpha_n^{2q-1} & \left[\mathrm{e}^{\mathrm{i}\alpha_n(x_j-1)} + (-1)^n \mathrm{e}^{-\mathrm{i}\alpha_n(x_j+1)} \right] \\ & \times \left[\mathrm{e}^{-\mathrm{i}\alpha_n(x-1)} + (-1)^{n+1} \mathrm{e}^{\mathrm{i}\alpha_n(x+1)} \right]. \end{aligned}$$

After some simplification, this reduces to

$$\overline{\phi_n^{(2q-1)}(x_j)} \phi_n(x) = 2|c_0|^2 \mathrm{i}^{2q-2} \alpha_n^{2q-1} \left[\sin \alpha_n(x - x_j) + (-1)^n \sin \alpha_n(x + x_j) \right].$$

Recall that $c^2 = 4|c_0|^2$. Upon substituting this into (4.1), and noticing that the term involving $(-1)^n \sin \alpha_n(x + x_j)$ is $\mathcal{O}\left(N^{-1}\right)$ (by Lemma 2), we obtain

$$f(x) - f_N(x) = \frac{1}{2}[f](x_j) \sum_{n>N} \frac{1}{\alpha_n} \sin \alpha_n(x - x_j) + \mathcal{O}\left(N^{-1}\right). \tag{4.2}$$

We are now able to prove the main result of this section:

Theorem 2. *For $j = 1, \ldots, k$ let U be a compact neighbourhood of x_j not containing x_l for $l \neq j$, $l = 0, \ldots, k + 1$. Suppose that*

$$\tilde{f}_N(x) = \frac{1}{2} \hat{f}_0^C + \sum_{n=1}^{N} \left[\hat{f}_n^C \cos n\pi x + \hat{f}_n^S \sin n\pi x \right],$$

is the truncated Fourier sum of f, where $\hat{f}_n^C = \int_{-1}^{1} f(x) \cos n\pi x \, \mathrm{d}x$ and $\hat{f}_n^S = \int_{-1}^{1} f(x) \sin n\pi x \, \mathrm{d}x$. Then

$$f_N(x) = g_1(x) \tilde{f}_{\frac{N}{2}}(x) + [1 - g_1(x)] f(x) - [f](x_j) g_2(x) \left[h(x) - \tilde{h}_{\frac{N}{2}}(x) \right] + \mathcal{O}\left(N^{-1}\right), \tag{4.3}$$

for $x \in U\backslash\{x_j\}$*, where*

$$g_1(x) = \cos \tfrac{1}{4}q\pi(x-x_j)\cos\tfrac{1}{2}\pi(x-x_j), \quad g_2(x) = \sin\tfrac{1}{4}q\pi(x-x_j)\cos\tfrac{1}{2}\pi(x-x_j),$$

and $h(x) = -\frac{1}{\pi}\log\left[2|\sin\frac{1}{2}\pi(x-x_j)|\right]$*. In particular,* $f_N(x) = \tilde{f}_{\frac{N}{2}}(x) + R(x)$*, where*

$$|R(x)| \leq c\left\{|x-x_j|^2 + |x-x_j||h(x) - \tilde{h}_{\frac{N}{2}}(x)| + N^{-1}\right\}, \quad x \in U\backslash\{x_j\},$$

and c *is a positive constant independent of* N *and* x*.*

Proof. First consider the Fourier sum of f. Since

$$\hat{f}_n^C = \int_{-1}^{1} f(x)\cos n\pi x \,\mathrm{d}x = -\frac{1}{n\pi}\sin n\pi x_j [f](x_j) + \mathcal{O}\left(n^{-2}\right),$$
$$\hat{f}_n^S = \int_{-1}^{1} f(x)\sin n\pi x \,\mathrm{d}x = \frac{1}{n\pi}\cos n\pi x_j [f](x_j) + \frac{(-1)^{n+1}}{n\pi}\left[f(1)-f(-1)\right] + \mathcal{O}\left(n^{-2}\right),$$

we find that

$$f(x) - \tilde{f}_N(x) = \sum_{n>N}\left\{\frac{1}{n\pi}\left[-\cos n\pi x\sin n\pi x_j + \sin n\pi x\cos n\pi x_j\right][f](x_j) \right.$$
$$\left. + \frac{(-1)^{n+1}}{n\pi}\left[f(1)-f(-1)\right]\right\} + \mathcal{O}\left(N^{-1}\right),$$

for $x \in U\backslash\{x_j\}$. The second term is $\mathcal{O}\left(N^{-1}\right)$. Hence, after simplifying, we obtain

$$f(x) - \tilde{f}_N(x) = [f](x_j)\sum_{n>N}\frac{1}{n\pi}\sin n\pi(x-x_j) + \mathcal{O}\left(N^{-1}\right), \quad x \in U\backslash\{x_j\}. \tag{4.4}$$

Now consider the polyharmonic–Dirichlet expansion. Using (4.2) and the fact that $\alpha_n \sim \frac{1}{2}n\pi + \frac{1}{4}(q-1)\pi$, we have

$$f(x) - f_N(x)$$
$$= \frac{1}{2}[f](x_j)\sum_{n>\frac{N}{2}}\frac{1}{n\pi}\left\{\sin[n\pi + \tfrac{1}{4}(q-1)\pi](x-x_j) + \sin[n\pi + \tfrac{1}{4}(q+1)\pi](x-x_j)\right\} + \mathcal{O}\left(N^{-1}\right).$$

Using (4.4), this now gives

$$f(x) - f_N(x) = g_1(x)\left[f(x) - \tilde{f}_{\frac{N}{2}}(x)\right] + g_2(x)[f](x_j)\sum_{n>\frac{N}{2}}\frac{1}{n\pi}\cos n\pi(x-x_j) + \mathcal{O}\left(N^{-1}\right).$$

Therefore, to prove (4.3), it suffices to show that

$$\int_{-1}^{1} h(x)\cos n\pi x\,\mathrm{d}x = \frac{1}{n\pi}. \tag{4.5}$$

The full result then follows immediately from periodicity and standard estimates. To establish (4.5), it is useful to introduce the Clausen function $\mathcal{C}_2(\theta)$ [1], given by

$$\mathcal{C}_2(\theta) = -\int_0^{\theta}\log|2\sin\tfrac{1}{2}t|\,\mathrm{d}t = \sum_{n=1}^{\infty}\frac{1}{n^2}\sin n\theta.$$

Note that the infinite sum on the right-hand side converges uniformly for θ in any compact subset of $\mathbb{R}$. Returning to (4.5), it is readily seen that $h(x) = \frac{1}{\pi^2}\frac{\mathrm{d}}{\mathrm{d}x}\mathcal{C}_2(\pi x)$. Hence, substituting this into (4.5) and integrating by parts, we obtain

$$\int_{-1}^{1} h(x)\cos n\pi x\,\mathrm{d}x = \frac{1}{\pi^2}\mathcal{C}_2(\pi x)\cos n\pi x\Big|_{-1}^{1} + \frac{n}{\pi}\int_{-1}^{1}\mathcal{C}_2(\pi x)\sin n\pi x\,\mathrm{d}x.$$

Since $\mathcal{C}_2$ has a uniformly convergent series expression, the result now follows immediately from othorgonality of the functions $\sin n\pi x$ on $[-1,1]$ and the fact that $\mathcal{C}_2(\pm\pi) = 0$. $\square$

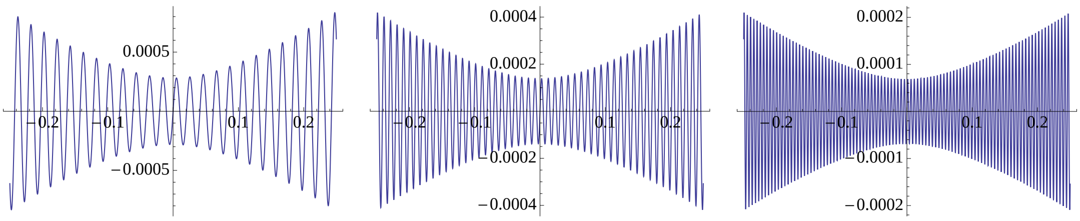

Figure 4: The error $f_N(x) - \tilde{f}_{\frac{N}{2}}(x)$ against $x \in [-\frac{1}{4}, \frac{1}{4}]$, where $q = 2$, $N = 100, 200, 400$ (left to right) and $f(x)$ is the Heavyside function.

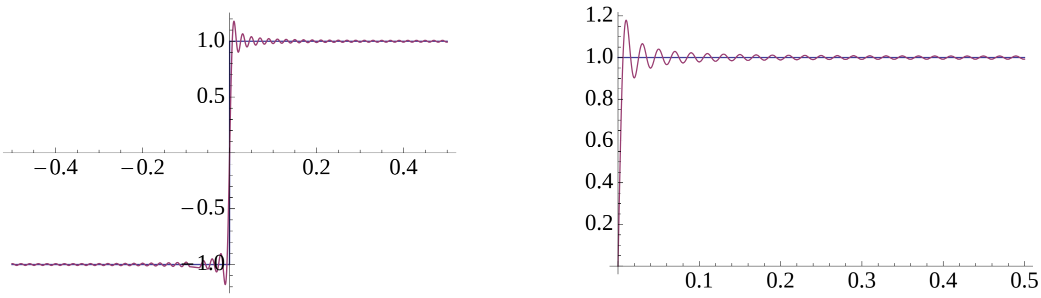

Figure 5: The error $f(x) - f_{100}(x)$ against $x \in [-\frac{1}{2}, \frac{1}{2}]$ (left) and $x \in [0, \frac{1}{2}]$ (right).

It is, at first, somewhat surprising that the expression (4.3) involves a term possessing a logarithmic singularity when the original function f has a jump discontinuity. Yet, this singularity is removable, since the function $g_2(x) = \mathcal{O}(x - x_j)$ for $x - x_j \ll 1$. Moreover, the appearance of Clausen functions is no great surprise given that, in general, the s^{th} such function, denoted $\mathcal{C}_s$, is equivalently defined as the unique odd function with n^{th} Fourier sine coefficient equal to n^{-s} [1]. Note that, although we have used such functions as a theoretical tool, their practical application to the removal of the Gibbs phenomenon in certain Fourier series has been considered in [11].

Returning to the problem at hand, recall the main conclusion of Theorem 2: polyharmonic–Dirichlet series are well approximated by Fourier series in neighbourhoods interior singularities. Closer inspection of the remainder term $R(x)$ confirms this fact. Indeed, if $x \in U$ is not within a distance of $\mathcal{O}\left(N^{-1}\right)$ of x_j, then $|R(x)| = \mathcal{O}\left(|x|N^{-1}\right)$. On the other hand, whenever $x - x_j = \mathcal{O}\left(N^{-1}\right)$, we have $|R(x)| = \mathcal{O}\left(N^{-1} \log N\right)$. In Figure 4 we confirm these estimates by plotting the error between the two truncated sums when $f(x)$ is the Heavyside function. Notice both the $\mathcal{O}\left(N^{-1}\right)$ decay and the linear growth in $|x|$ away from the discontinuity at $x = 0$, as predicted.

As a simple consequence of Theorem 4, we may now precisely determine the maximal overshoot of polyharmonic–Dirichlet expansions:

Corollary 1. *Let f have an interior jump discontinuity at x_j. Then, for sufficiently large N, the truncated polyharmonic–Dirichlet expansion f_N has maximal overshoot in a neighbourhood of x_j occurring at $x_j + \frac{2}{N}$. Moreover, $f_N(x_j) \to [f](x_j)$ as $N \to \infty$ and*

$$f_N(x_j \pm \tfrac{2}{N}) = f(x_j^{\pm}) \pm c^*[f](x_j) + \mathcal{O}\left(N^{-1}\right),$$

where $c^* = \frac{1}{\pi}\int_0^{\pi} \frac{\sin x}{x}\,\mathrm{d}x - \frac{1}{2} \approx 0.08949$.

This corollary confirms that the interior Gibbs phenomenon for polyharmonic–Dirichlet expansions is identical to that of Fourier series. In Figure 5 we highlight this result for the case $q = 2$. Note that the maximal overshoot value, as predicted by Corollary 1 and corroborated by this figure, is $1 + 2c^* \approx 1.17898$ in this case.

4.2 Endpoint Gibbs phenomenon

As discussed, polyharmonic–Dirichlet eigenfunctions cease to behave like regular oscillators as x approaches ± 1. As a result, polyharmonic–Dirichlet expansions no longer resemble classical Fourier series in intervals containing either endpoint. Nonetheless, we are still able to determine the maximal overshoot near $x = \pm 1$, thereby quantifying the Gibbs phenomenon in this case

Assume that U is a compact neighbourhood of $x = 1$ (the case $x = -1$ is virtually identical), with $x_j \notin U$ for $j = 0, \ldots, k$. As before, we may write

$$f(x) - f_N(x) = \frac{(-1)^q \overline{d_{2q-1} f}(1)}{c^2} \sum_{n>N} \frac{1}{\alpha_n} \phi_n(x) + \mathcal{O}\left(N^{-1}\right), \quad x \in U \backslash \{x_j\}.$$

Let $x = 1 - \frac{2a}{N}$, where $a > 0$ is fixed. Substituting the asymptotic estimate (2.4), we obtain

$$\begin{aligned} & f(1 - \tfrac{2a}{N}) - f_N(1 - \tfrac{2a}{N}) \\ & \quad = \frac{(-1)^q \overline{d_{2q-1} f}(1)}{c^2} \sum_{n>N} \frac{1}{\alpha_n} \left\{ c_0 \left[\mathrm{e}^{\mathrm{i} \frac{2a\alpha_n}{N}} - \mathrm{i}^{q-1} \mathrm{e}^{-\mathrm{i} \frac{2a\alpha_n}{N}} \right] + \sum_{r=1}^{q-1} c_r \mathrm{e}^{-\lambda_r \frac{2a\alpha_n}{N}} \right\} + \mathcal{O}\left(N^{-1}\right). \end{aligned}$$

Hence,

$$\begin{aligned} & f(1 - \tfrac{2a}{N}) - f_N(1 - \tfrac{2a}{N}) \\ & \quad = \frac{2(-1)^q \overline{d_{2q-1} f}(1)}{\pi c^2} \int_{\pi}^{\infty} \frac{1}{x} \left\{ c_0 \left[\mathrm{e}^{\mathrm{i}ax} - \mathrm{i}^{q-1} \mathrm{e}^{-\mathrm{i}ax} \right] + \sum_{r=1}^{q-1} c_r \mathrm{e}^{-\lambda_r ax} \right\} \mathrm{d}x + \mathcal{O}\left(N^{-1}\right). \end{aligned}$$

This now gives $f_N(1 - \frac{2a}{N}) = [1 + G(a)] f(1) + \mathcal{O}\left(N^{-1}\right)$, where

$$\begin{aligned} G(a) &= \frac{2(-1)^{q+1} \overline{d_{2q-1}}}{\pi c^2} \int_{\pi}^{\infty} \frac{1}{x} \left\{ c_0 \left[\mathrm{e}^{\mathrm{i}ax} - \mathrm{i}^{q-1} \mathrm{e}^{-\mathrm{i}ax} \right] + \sum_{r=1}^{q-1} c_r \mathrm{e}^{-\lambda_r ax} \right\} \mathrm{d}x \\ &= \frac{2(-1)^{q+1} \overline{d_{2q-1}}}{\pi c^2} \left\{ c_0 \left[\Gamma(0, -\mathrm{i}a\pi) - \mathrm{i}^{q-1} \Gamma(0, \mathrm{i}a\pi) \right] + \sum_{r=1}^{q-1} c_r \Gamma(0, \lambda_r \pi a) \right\}, \end{aligned}$$

and $\Gamma(\cdot, \cdot)$ is the incomplete gamma function [1]. From this we conclude

Theorem 3. *For sufficiently large N, f_N has maximal overshoot in an $\mathcal{O}\left(N^{-1}\right)$ neighbourhood of the endpoint $x = 1$ occurring at $x = 1 - \frac{2a^*}{N}$, where $a^* = \mathrm{argmax}_{a \geq 0} G(a)$. In addition,*

$$\limsup_{\substack{x \to 1^- \\ N \to \infty}} f_N(x) = \lim_{N \to \infty} f_N(1 - \tfrac{2a^*}{N}) = [1 + G(a^*)] f(1) > f(1).$$

Aside from the $q = 1$ case, where $a^* = 1$, the value a^* must be found numerically. Note that a^* is a zero of the function

$$G'(a) = -\frac{1}{a} \left\{ c_0 \mathrm{i}^{q-1} \mathrm{e}^{-\mathrm{i}\pi a} + \sum_{r=1}^{q-1} c_r \mathrm{e}^{-\lambda_r a\pi} \right\}.$$

In Table 1 we report the value of a^* and $G(a^*)$ for various q. Additionally, Figure 6 gives a plot of $f_N(x)$ near $x = 1$, confirming these results. Notice that, as q increases, so does the value a^*. Thus the overshoot moves further away from the endpoint $x = 1$.

If required, we could also compute the successive local maxima and minima of $G(a)$. As shown in Figure 6, these correspond to overshoots and undershoots of the approximation of f by f_N. In the Fourier case, these occur precisely at the values $x = 1 - \frac{2k-1}{N}$ and $x = 1 - \frac{2k}{N}$, $k \in \mathbb{N}_+$, respectively. Though this is not true in the polyharmonic–Dirichlet setting, the exponential decay (as a increases) of the terms $\mathrm{e}^{-\lambda_r a\pi}$ appearing in $G'(a)$, indicates that successive maxima and minima will become increasingly equispaced away from $x = 1$, as in the Fourier scenario.

	$q=1$	$q=2$	$q=3$	$q=4$
a^*	1	1.25437	1.52315	1.74643
$1+G(a^*)$	1.1798	1.20705	1.21958	1.22792

Table 1: The values a^* and $1+G(a^*)$ for $q=1,2,3,4$.

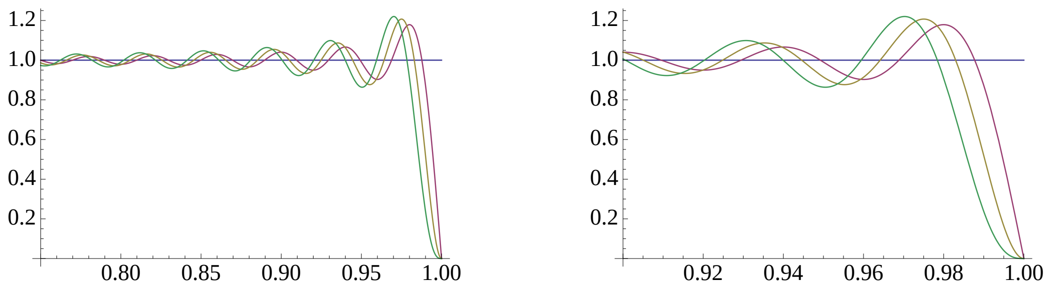

Figure 6: The function $f_{50}(x)$ for $x \in [\frac{3}{4}, 1]$ (left) and $x \in [\frac{9}{10}, 1]$ and $q=1,2,3$, where $f(x)=1$.

Another consequence of Theorem 3 is that, unlike the interior case, the endpoint Gibbs phenomenon is local: it involves only the value of f at $x=1$. Furthermore, since polyharmonic–Dirichlet eigenfunctions vanish at $x=1$, so does the truncated expansion f_N (see Figure 6). In contrast, periodicity ensures that the Fourier series Gibbs phenomenon is identical regardless of where the discontinuity is located in $[-1,1]$.

In the next section we address the removal of the Gibbs phenomenon. First, however, we remark that, as consequence of this and the previous section, we have actually proved a version of the well-known Dirichlet–Jordan theorem [38] for polyharmonic–Dirichlet series. We have

Theorem 4. *The truncated polyharmonic–Dirichlet expansion of f_N converges pointwise to $f(x)$ whenever f is continuous at $x \in (-1,1)$. If $x = \pm 1$, then $f_N(x) = 0$ for all N, and if f has a jump discontinuity at x, then $f_N(x) \to \frac{1}{2}[f(x^+) + f(x^-)]$ as $N \to \infty$.*

Once more, we observe the distinct character of the endpoints $x = \pm 1$. Note that we have deliberately presented this theorem without specific regularity assumptions (we imposed the overly cautious condition of piecewise analyticity throughout). It remains to be seen whether similar conditions can be imposed to those found in the Fourier case, i.e. bounded variation [38].

5 Removal of the Gibbs phenomenon

Having detailed the Gibbs phenomenon for polyharmonic–Dirichlet expansions, we now develop a number of techniques to first ameliorate and then completely remove this effect. By the former we mean that, given the first N polyharmonic–Dirichlet coefficients of a function f, we seek a new approximation $\bar{f}_N$ that suffers from the Gibbs phenomenon only in some higher derivative r, say, and correspondingly delivers uniform convergence at the increased rate of N^{-r}. Similarly, for the complete removal of the Gibbs phenomenon, we desire an approximation in which no derivative suffers from this phenomenon, and which possesses spectral convergence (i.e. faster than any algebraic power of N^{-1}).

As discussed, whenever f has no interior discontinuities, a simple approach to mitigate the Gibbs phenomenon is to replace Dirichlet eigenfunctions with their Neumann counterparts (1.2). Yet, whilst this attains uniform convergence of the expansion, it only realises a convergence rate of finite, fixed algebraic order. In fact, $\|f - f_N\|_\infty = \mathcal{O}\left(N^{-q}\right)$, where f_N is the polyharmonic–Neumann expansion of f and $\|\cdot\|_\infty$ is the uniform norm on $[-1,1]$ [5].

In light of this fact, we shall remain in the polyharmonic–Dirichlet setting. Thus, we assume that the first N polyharmonic–Dirichlet coefficients $\hat{f}_n$ have been computed, and we now wish to obtain a more rapidly convergent approximation to f than the projection f_N. For simplicity, we shall assume that f is analytic on $[-1,1]$. In particular, f possesses no jump discontinuities in $(-1,1)$. The extension of techniques developed herein to the general case is conceptually simple, the one caveat being that the location of the discontinuities must be known (in the language of signal processing, we consider only the *reconstruction* problem. Clearly a complete method must also include a procedure for singularity *detection*. In the context of Fourier series, with potential for extension to this case, a number of such methods exist [36]).

Our starting point is the asymptotic expansion (3.6) for the error $f(x) - f_N(x)$. Since f has no interior discontinuities, this reads

$$f(x) - f_N(x) \\ \sim \sum_{r=0}^{\infty} \sum_{s=0}^{q-1} (-1)^{(r+1)q+s} \Big\{ f^{(2rq+s)}(1) \Theta_N(1,r,s;x) - f^{(2rq+s)}(-1) \Theta_N(0,r,s;x) \Big\}, \tag{5.1}$$

Recall from Lemma 1 that the functions $\Theta_N(j,r,s;x)$ are $\mathcal{O}\left(N^{-2rq-s-1}\right)$ for $x \neq x_j$. In fact, with a little effort it can be shown that $\|\Theta_N(j,r,s;\cdot)\|_\infty = \mathcal{O}\left(N^{-2rq-s}\right)$. With this in mind, (5.1) provides an important observation: the derivatives $f^{(2rq+s)}(\pm 1)$ completely determine the rate of convergence of f_N. Had such derivatives vanished (up to a certain order), faster convergence would have been witnessed. Specifically, suppose that $f^{(l)}(\pm 1) = 0$ whenever $l \in D_{k,p}$, where

$$D_{k,p} = \{ l \in \mathbb{N} : l = 2rq + s < 2kq + p, \ r \in \mathbb{N}, \ s = 0, \ldots, q-1 \}, \quad k \in \mathbb{N}, \ p = 0, \ldots, q-1,$$

then the expansion f_N satisfies the error estimates

$$\|f - f_N\|_\infty = \mathcal{O}\left(N^{-2kq-p}\right), \quad f(x) - f_N(x) = \mathcal{O}\left(N^{-2kq-p-1}\right), \quad x \in (-1,1). \tag{5.2}$$

Hence, faster, and indeed uniform, convergence. Note that these derivative conditions can be viewed as the natural analogue of periodicity for polyharmonic–Dirichlet expansions (in particular, had we been concerned with regularity in this paper, we could have introduced an analogue of the periodic Sobolev spaces $\mathrm{H}^k(\mathbb{T})$ for polyharmonic–Dirichlet expansions using such conditions).

Unfortunately, the assumption of vanishing derivatives is unrealistic. However, with the understanding that it is those derivatives with indices in $D_{k,p}$ which determine the convergence rate, we are able to develop a simple technique to obtain faster convergence. This approach is a generalisation of a well-known method in the context of Fourier series: namely, the *polynomial subtraction* device [19, 27] (also known as *Krylov's method* [25] or the *Bernoulli method* [18]).

5.1 Polynomial subtraction

Suppose that g satisfies

$$g^{(l)}(\pm 1) = f^{(l)}(\pm 1), \quad \forall l \in D_{k,p}. \tag{5.3}$$

Then the $(k,p)^{\text{th}}$ polynomial subtraction approximation $\bar{f}_{N,k,p}$ is defined by $\bar{f}_{N,k,p} = (f_N - g_N) + g$. Since the error $f - f_{N,k,p} = (f-g) - (f-g)_N$, and $f - g$ has vanishing derivatives with indices $l \in D_{k,p}$, we immediately see that the approximation $f_{N,k,p}$ obtains the faster convergence rates given by (5.2). Thus, by choosing k, p suitably, we can obtain algebraic convergence in N of any fixed order. As a result, the Gibbs phenomenon can be ameliorated. In fact, though we shall not show this, it is only appears in the derivative $(f_N)^{(2kq+p)}$. Additionally, those derivatives $(f_N)^{(l)}$ with $l < 2kq + p$ converge uniformly to the corresponding derivatives of f.

The main question remaining is how to construct the function g. Typically, this is achieved with a polynomial (hence the name *polynomial* subtraction). For $q = 1$ it is well-known (see [8, 32]) that such a function g has the explicit representation

$$g(x) = \sum_{r=0}^{k-1} 2^{2r} \left[\Lambda_r\left(\tfrac{1+x}{2}\right) f^{(2r)}(1) + \Lambda_r\left(\tfrac{1-x}{2}\right) f^{(2r)}(-1) \right], \tag{5.4}$$

where $\Lambda_r \in \mathbb{P}_{2r+1}$ is the r^{th} Lidstone polynomial [7], defined by $\Lambda_0 = x$ and

$$\Lambda_r'' = \Lambda_{r-1}, \quad \Lambda_r(0) = \Lambda_r(1) = 0, \quad r = 1, 2, \ldots. \tag{5.5}$$

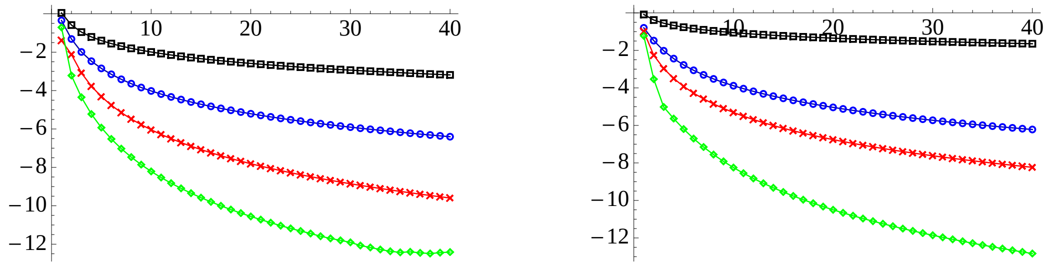

Figure 7: Error in polynomial subtraction applied to $f(x) = \mathrm{e}^x \cos 4x$. (left) Log error $\log_{10} \|f - \bar{f}_{N,k}\|_\infty$ against N for $q = 1$ with $k = 1, 2, 3, 4$ (in descending order). (right) error $\log_{10} \|f - \bar{f}_{N,k,p}\|_\infty$ for $q = 2$ with $(k, p) = (0, 0), (0, 1), (1, 0), (1, 1)$.

Note that g, as given by (5.4), is a polynomial of degree $2k-1$ and is the unique Birkhoff–Hermite interpolating polynomial satisfying $g^{(2r)}(\pm 1) = f^{(2r)}(\pm 1)$, $r = 0, \ldots, k-1$. We mention in passing that Birkhoff–Hermite problems (interpolation problems based on lacunary derivatives) need not have solutions in general (unlike pure Hermite problems) [26]. However, in this case, as evidenced by (5.4), the problem is uniquely solvable.

Let us now consider the general setting $q \geq 1$. Given f, we seek a function g that satisfies the interpolation conditions (5.3). Notice that the Lidstone polynomials (5.5) are defined as solutions of Poisson's equation. This suggests the following generalisation. For $r = 0, \ldots, q-1$ define $\Lambda_r \in \mathbb{P}_{2r+1}$ by

$$\Lambda_r^{(s)}(0) = \Lambda_r^{(s)}(1) = 0, \quad s = 0, \ldots, r-1, \quad \Lambda_r^{(r)}(0) = 0,\ \Lambda_r^{(r)}(1) = 1, \tag{5.6}$$

and, for arbitrary $r \geq q$, let $\Lambda_r \in \mathbb{P}_{2r+1}$ be given by

$$\Lambda_r^{(2q)} = \Lambda_{r-q}, \quad \Lambda_r^{(s)}(0) = \Lambda_r^{(s)}(1) = 0, \quad s = 0, \ldots, q-1. \tag{5.7}$$

We refer to $\{\Lambda_r\}_{r=1}^{\infty}$ as *q-Lidstone* polynomials. Note that the existence and uniqueness of such polynomials is an immediate consequence of the positive definiteness of the polyharmonic–Dirichlet operator and standard results regarding Hermite interpolation. In addition, it is also simple to confirm that the polynomial $\Lambda_{rq+s}\left(\frac{1 \pm x}{2}\right)$ has polyharmonic–Dirichlet coefficient

$$\frac{(-1)^{(r+1)q+s}}{\alpha_n^{2(r+1)q}} \phi^{(2q-s-1)}(\pm 1), \quad n = 1, 2, \ldots.$$

Returning to the construction of g, we have

Lemma 4. *The polynomial*

$$\begin{aligned} g(x) = &\sum_{r=0}^{k-1} \sum_{s=0}^{q-1} 2^{2rq+s} \left[\Lambda_{rq+s}\left(\tfrac{1+x}{2}\right) f^{(2rq+s)}(1) + (-1)^s \Lambda_{rq+s}\left(\tfrac{1-x}{2}\right) f^{(2rq+s)}(-1) \right] \\ &+ \sum_{s=0}^{p-1} 2^{2rq+s} \left[\Lambda_{kq+s}\left(\tfrac{1+x}{2}\right) f^{(2kq+s)}(1) + (-1)^s \Lambda_{kq+s}\left(\tfrac{1-x}{2}\right) f^{(2kq+s)}(-1) \right], \end{aligned}$$

is the unique polynomial of degree $2(kq+p)-1$ *satisfying (5.3).*

Proof. This follows immediately from the definition of the polynomials Λ_r. □

In Figure 7 we demonstrate polynomial subtraction for $q = 1, 2$. Note the higher accuracy gained from increasing the degree of the subtraction polynomial g. In particular, using only $N = 40$ and $(k, p) = (1, 1)$ (when $q = 2$), we obtain 12 digits of accuracy.

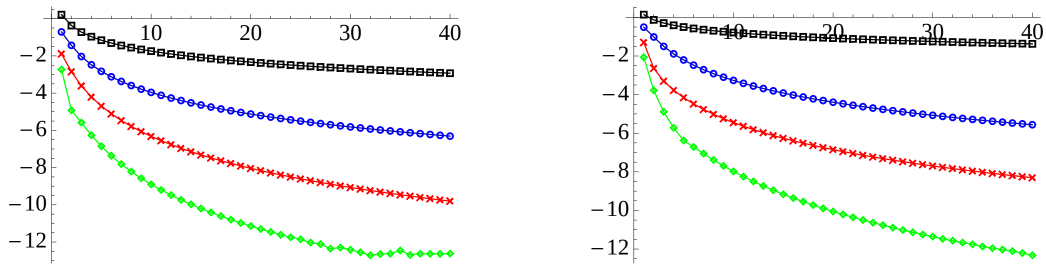

Figure 8: Error in Eckhoff's method applied to $f(x) = \mathrm{e}^x \cos 4x$. (left) Log error $\log_{10} \|f - \bar{f}_{N,k}\|_\infty$ against N for $q = 1$ with $k = 1, 2, 3, 4$ (in descending order). (right) error $\log_{10} \|f - \bar{f}_{N,k,p}\|_\infty$ for $q = 2$ with $k, p = (0, 0), (0, 1), (1, 0), (1, 1)$.

5.2 Extrapolation-based techniques

The polynomial subtraction device is widely used in the context of Fourier series. As considered, once the particular factors that determine the convergence rate of polyharmonic–Neumann expansions are understood, it can be readily generalised to this setting. Unfortunately, this technique suffers from the restriction of requiring exact derivative values. In general these are not readily available, and approximation via finite differences is not recommended for this task [27]. Fortunately, for Fourier series at least, a technique to circumvent this problem is also known. This approach, referred to as Eckhoff's method [15, 16], is based on the idea that the coefficients $\hat{f}_n$ themselves contain sufficient information to approximate such derivative values.

Eckhoff's method can be extended to polyharmonic–Dirichlet expansions in a straightforward manner. The starting point is the asymptotic expansion (3.2) for the coefficient $\hat{f}_n$. We have

$$\hat{f}_n \sim \sum_{r=0}^{\infty} \sum_{s=0}^{q-1} \frac{(-1)^{(r+1)q+s}}{\alpha_n^{2(r+1)q}} \left[f^{(2rq+s)}(1) \overline{\phi_n^{(2q-s-1)}(1)} - f^{(2rq+s)}(-1) \overline{\phi_n^{(2q-s-1)}(-1)} \right].$$

Suppose now that the function g interpolates exactly those derivatives $f^{(l)}(\pm 1)$ with $l \in D_{k,p}$. Then, it is readily seen that $\hat{f}_n = \hat{g}_n + \mathcal{O}\left(n^{-2kq-p-1}\right)$. To avoid the use of derivatives in the construction of the function g, we enforce this relation in the asymptotic limit $n \to \infty$. To do so, we define the new function g by

$$\hat{f}_n = \hat{g}_n, \quad n = N+1, N+2, \ldots, N+2(kq+p), \tag{5.8}$$

a $(2kq + 2p) \times (2kq + 2p)$ linear system for the coefficients of g. As before, we introduce the new approximation via $\bar{f}_{N,k,p} = (f_N - g_N) + g$. Since this procedure is reminiscent of (but not identical to) the Richardson extrapolation method [33], we refer to it as an *extrapolation-based* technique.

When $q = 1$, this method has been thoroughly studied in [3]. In fact, it has been shown that this process does not lead to a deterioration in the convergence rate over polynomial subtraction. In particular, the uniform error $\|f - \bar{f}_{N,k,p}\|_\infty$ remains $\mathcal{O}\left(N^{-2kq-p}\right)$. Thus, exact derivatives are not necessary to obtain faster convergence of polyharmonic–Dirichlet expansions.

The main drawback of this device is that the linear system to be solved is extremely ill-conditioned. Nonetheless, as discussed in [3], there are a number of ways to mitigate this effect. First, we replace the linear system (5.8) with an overdetermined least squares problem. Second, instead of forming g as a linear combination of q-Lidstone polynomials, we employ a set consisting of, for example, Chebyshev or Legendre polynomials (nonpolynomial choices, such as trigonometric functions, also confer a similar benefit [3]). In Figure 8 we give numerical results for Eckhoff's method applied to the function $f(x) = \mathrm{e}^x \cos 4x$. Upon comparison with Figure 7, we notice that the ill-conditioning has little effect on the resultant approximation. Furthermore, as previously commented, there is no deterioration in the convergence rate.

Whilst Figure 8 confirms that the approximation $\bar{f}_{N,k,p}$ performs as expected, we shall not provide any analysis of Eckhoff's method in this setting. Instead, we now detail an approach to completely remove the Gibbs phenomenon (as opposed to ameliorating it to a certain order). As we prove, the resulting method delivers spectral accuracy.

5.3 Least squares methods

An alternative to extrapolation techniques is to augment the approximation space suitably and use a least squares criterion to compute the approximation. Suppose that we consider the system $\mathcal{H} = \{\phi_n : n \in \mathbb{N}_+\} \cup \{\psi_n : n \in \mathbb{N}_+\}$ consisting of both polyharmonic–Dirichlet and polyharmonic–Neumann eigenfunctions, denoted by ϕ_n and ψ_n respectively. If $\mathcal{H}_N$ is the finite subset $\{\phi_n : n = 1, \ldots, N\} \cup \{\psi_n : n = 1, \ldots, N\}$, we seek an approximation

$$\bar{f}_N(x) = \sum_{n=1}^{N} [a_n \phi_n(x) + b_n \psi_n(x)] \in \mathrm{span}\mathcal{H}_N,$$

defined by the least squares criterion

$$\bar{f}_N = \arg\min_{g \in H_N} \|f - g\|. \tag{5.9}$$

In matrix form, the coefficients a_n, b_n of the function f_N are computed by solving the least squares problem $Ax = y$, where

$$A = \begin{pmatrix} I & C \\ C^\top & I \end{pmatrix}, \quad x = (a_1, \ldots, a_N, b_1, \ldots, b_N)^\top, \quad y = (\hat{f}_1, \ldots, \hat{f}_N, \check{f}_1, \ldots, \check{f}_N)^\top,$$

$\check{f}_n = \int_{-1}^{1} f(x)\psi_n(x)\,\mathrm{d}x$ and $C \in \mathbb{R}^{N \times N}$ has $(n, m)^{\text{th}}$ entry $\int_{-1}^{1} \phi_n(x)\psi_m(x)\,\mathrm{d}x$.

As with Eckhoff's method, ill-conditioning also occurs with this approach. Hence, we typically overdetermine the problem in practice. This corresponds to replacing the square matrix A with an augmented $2M \times 2N$ matrix and the vector y with a vector of length $2M$ (here $M \geq N$).

This issue aside, however, we can now prove spectral convergence of the approximation $\bar{f}_N$, and thus the confirm the complete removal of the Gibbs phenomenon by this approach. We have

Theorem 5. *The approximation $\bar{f}_N$ converges spectrally fast to f. In particular, $\|f - \bar{f}_N\| \leq c_k(f)N^{-2kq}$, $\forall k \in \mathbb{N}$, for some positive constant $c_k(f)$ depending only on f and k.*

Proof. Since $\bar{f}_N$ is defined by (5.9), we have

$$\|f - \bar{f}_N\| \leq \|f - h_N\|, \quad \forall h_N \in \mathrm{span}\mathcal{H}_N. \tag{5.10}$$

Let $N > 2kq$. Suppose that we can find a function $\psi \in \mathrm{span}\{\psi_1, \ldots, \psi_N\}$ such that

$$\psi^{(2rq+s)}(\pm 1) = f^{(2rq+s)}(\pm 1), \quad r = 0, \ldots, k-1, \quad s = 0, \ldots, q-1.$$

Then, letting $h_N = f_N - \psi_N + \psi$ in (5.10), where f_N and ψ_N are the expansions of f and ψ in polyharmonic–Dirichlet eigenfunctions respectively (note that $h_N \in \mathrm{span}\mathcal{H}_N$), the result now follows immediately from the arguments of Section 5.1, since h_N is a polynomial subtraction approximation to f (albeit one formed with a nonpolynomial subtraction function).

Hence, to complete the proof we wish to show that it is always possible to find such a function ψ. Suppose that $M > 0$ and that $2kq + 2M \leq N$. Set

$$\psi(x) = \sum_{n=2M}^{2kq+2M-1} a_n \psi_n(x). \tag{5.11}$$

We claim that, for sufficiently large M, it is alway possible to find a function ψ of this form satisfying $\psi^{(2rq+s)}(\pm 1) = c^{\pm}_{rq+s}$, $r = 0, \ldots, k-1$, $s = 0, \ldots, q-1$, for arbitrary constants $c^{\pm}_{rq+s}$.

To establish this claim, we first note that virtually identical exponential asymptotics hold for polyharmonic–Neumann eigenfunctions ψ_n as those detailed in Section 2 for the Dirichlet case (see also [5]). In particular, $\psi_n^{(r)}(\pm 1) = \alpha_n^r d_r (\pm 1)^{r+n+q+1} + \mathcal{O}\left(n^r \mathrm{e}^{-n\pi\gamma_q}\right)$. Hence,

$$\sum_{n=2M}^{2kq+2M-1} a_n \left[\alpha_n^{2rq+s}(\pm 1)^n + E^{\pm}_{rq+s,n}\right] = \frac{(\pm 1)^{r+q+1}}{d_{2rq+s}} c^{\pm}_{2rq+s}, \quad r = 0, \ldots, k-1, \; s = 0, \ldots, q-1,$$

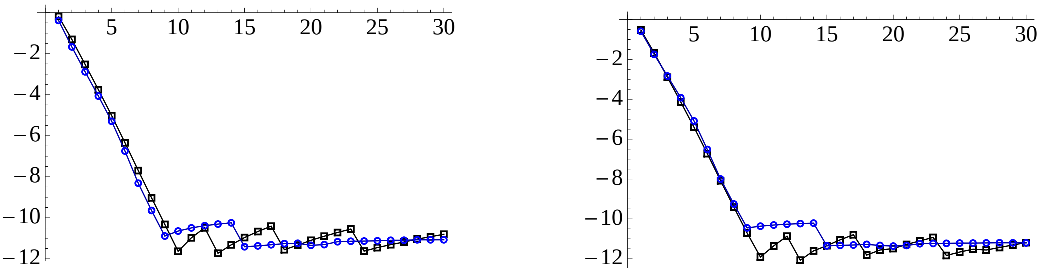

Figure 9: Log error $\log_{10} \|f - \bar{f}_N\|_\infty$ against $N = 1, \ldots, 30$ for $q = 1$ (squares) and $q = 2$ (circles), where $f(x) = \cosh x$ (left) and $f(x) = x \cos(x + 1)$ (right).

where $E^{\pm}_{rq+s,n} = \mathcal{O}\left(n^{2rq+s} \mathrm{e}^{-n\pi\gamma_q}\right)$. After separating terms corresponding to $(\pm 1)^n$, we find that

$$\sum_{n=M}^{kq+M-1} a_{2n} \left[\alpha_{2n}^{2rq+s} + E_{rq+s,2n}\right] = C_{rq+s}, \qquad \sum_{n=M}^{kq+M-1} a_{2n+1} \left[\alpha_{2n+1}^{2rq+s} + E_{rq+s,2n+1}\right] = D_{rq+s},$$

for arbitrary values C_{2rq+s} and D_{2rq+s}, where $E_{rq+s,n} = \mathcal{O}\left(n^{2rq+s} \mathrm{e}^{-n\pi\gamma_q}\right)$. Consider the first system of equations. The claim is now seen to hold, provided the matrix with entries α_{2n}^{2rq+s} is nonsingular and has condition number growing only algebraically with M. Moreover, since $\alpha_n = \mathcal{O}(n)$, it is trivial to see that the condition number must be only at worst algebraically large in M. Hence, we need only show that this matrix is nonsingular.

Consider the transpose of this matrix. Seeking a contradiction, we assume that

$$\sum_{r=0}^{k-1} \sum_{s=0}^{q-1} b_{rq+s} \alpha_{2(n+M)}^{2rq+s} = 0, \quad n = 0, \ldots, kq - 1.$$

Let $P(x)$ be the polynomial $\sum_{r=0}^{k-1} \sum_{s=0}^{q-1} b_{rq+s} x^{2rq+s}$, so that $P(\alpha_{2(n+M)}) = 0$ for $n = 0, \ldots, kq - 1$. We claim that P must be identically zero.

To establish this claim, we use induction on k. For $k = 1$, $P(x) = \sum_{s=0}^{q-1} b_s x^s$, and the result follows immediately. Now assume that the result holds up to and including k. Define P as above, with k replaced by $k+1$, and assume that $P(x)$ vanishes at $x = \alpha_{2(n+M)}$, $n = 0, \ldots, (k+1)q - 1$. A simple argument concludes that the q^{th} derivative $P^{(q)}$ has at least kq simple zeros in the region $[\alpha_{2M}, \infty)$. However,

$$P^{(q)}(x) = x^q \sum_{r=0}^{k-1} \sum_{s=0}^{q-1} \tilde{b}_{rq+s} x^{2rq+s} = x^q Q(x),$$

for some constants $\tilde{b}_{rq+s}$. It follows that the Q must have at least kq simple zeros in $[\alpha_{2M}, \infty)$. However $Q \equiv 0$ by induction, and thus $P \equiv 0$, therefore completing the proof. $\square$

In Figure 9 we present numerical results for this method in the cases $q = 1$ and $q = 2$. As predicted, spectral convergence occurs. Indeed, these examples indicate that the approximation actually converges exponentially fast; an observation which, as we next discuss, has been confirmed in the $q = 1$ case.

This method can be viewed as a generalisation of the Fourier extension method [10, 22] to arbitrary $q \geq 1$. Indeed, the $q = 1$ case corresponds precisely to this method. As the name suggests, the Fourier extension method is intimately related to Fourier series. In fact, the approximation $\bar{f}_N$, being of the form

$$\bar{f}_N(x) = a_0 + \sum_{n=1}^{N} \left[a_n \cos \tfrac{1}{2} n\pi x + b_n \sin \tfrac{1}{2} n\pi x\right], \tag{5.12}$$

N	1	2	4	8	16
$\|u-\bar{u}_N\|_\infty$	1.56×10^{-3}	3.70×10^{-5}	2.08×10^{-8}	1.68×10^{-14}	9.85×10^{-15}

Table 2: Error in approximating u, where $u^{(4)}(x)+u(x)=4+x$, $u(\pm1)=u'(\pm1)=0$, by $\bar{u}_N$.

is readily identified as a truncated Fourier series on the extended domain $[-2,2]$. Thus, this procedure numerically computes a smooth periodic extension of the original function f on $[-2,2]$. In light of standard approximation properties of Fourier series of periodic functions, spectral convergence is therefore expected.

The Fourier extension method has been thoroughly analysed in [22]. The principal result confirms exponential convergence in N (for analytic functions f) at a rate of E^{-N}, where $E\approx 5.828$. Unfortunately, when $q\geq 2$ the analogy with Fourier series is lost. However, we are still able to verify spectral convergence in this case (Theorem 5), and therefore the removal of the Gibbs phenomenon.

Note that this approach requires both the coefficients $\hat{f}_n$ and $\check{f}_n$ to be known (or computed) explicitly. However, a relatively minor adjustment can be made to tackle the case where only the polyharmonic–Dirichlet coefficients $\hat{f}_n$ are given. In this case, we solve the linear system $Ax=y$, where

$$A=\begin{pmatrix} I & C \\ 0 & D\end{pmatrix},\quad y=(\hat{f}_1,\ldots,\hat{f}_{2N})^\top,$$

and $D_{n,m}=\int_{-1}^{1}\phi_{n+N}(x)\psi_m(x)\,\mathrm{d}x$. Note that the resultant approximation is no longer the solution of the least squares problem (5.9). Nonetheless, although we shall not prove it, this scheme also converges spectrally fast.

Conclusions

The intent of this paper was to describe the Gibbs phenomenon in polyharmonic–Dirichlet expansions and consider techniques for its removal. In particular, we have shown that the Gibbs phenomenon is identical at internal singularities to that occurring in standard Fourier series, whereas near the endpoints the phenomenon has a different character. Next, we developed technique for removal of this phenomenon, culminating in a method which delivered spectral accuracy using combinations of polyharmonic–Dirichlet and polyharmonic–Neumann eigenfunctions.

Potential applications of this work are the subject of current investigations. One obvious application is the numerical solution of fourth and higher-order boundary value problems. For example, if u is the solution of the biharmonic problem $u^{(4)}(x)+bu(x)=f(x)$, $u(\pm1)=u'(\pm1)=0$, where $b>0$, then u can be immediately expanded in its biharmonic–Dirichlet series. Indeed, the n^{th} biharmonic–Dirichlet coefficient of u is precisely $\hat{u}_n=(b+\alpha_n^4)^{-1}\hat{f}_n$. With this observation to hand, we can immediately apply the technique of Section 5.3 (for example) to compute an approximation $\bar{u}_N$ to u. In Table 2 we provide numerical results for the example with $f(x)=4+x$ and $b=1$. Using only $N=8$ (thus an approximation comprising 16 terms) we obtain an error of order 10^{-14}. Encouraged by this particular example, future work will address the application of the approach to a broader variety of problems.

Nonetheless, it seems preferable to use a small value of q (most likely $q=1$), unless the particular problem at hand lends itself naturally to a specific value (e.g. solving the aforementioned boundary value problem). As discussed in [5, 6], complications arise for larger q (computational cost and round-off error). In addition, the examples in Figures 7–9 indicate that there is no advantage gained in general from larger values of q. However, even when $q=1$ there remain a number open problems. In particular, all known techniques to remove the Gibbs phenomenon from Fourier (or Fourier-like) series suffer from ill-conditioning. A theoretical justification of this observation has been established in [31]: any exponentially convergent scheme based on Fourier coefficients must possess exponentially poor conditioning. However, there may be ways to circumvent this issue if the condition of exponential convergence was sufficiently relaxed.

Outside the issue its removal, it is of independent theoretical interest that the Gibbs phenomenon can be so accurately described in both this and many other instances (see Section

1). Even when there is no obvious connection to Fourier series, we still observe a similar phenomenon. A natural question to ask is whether the work of this paper can be generalised to even larger families of eigenfunction expansions. For example, polyharmonic–Dirichlet expansions can be viewed as a particular type of so-called *Birkhoff series* [6, 9, 29]. Despite lacking the exponential asymptotics of the polyharmonic case [5], it may be possible to extend the results of this paper to this setting.

A final topic for future investigation involves the q-Lidstone polynomials introduced in Section 5.1. Standard Lidstone polynomials (and, more generally, Lidstone series) have been extensively studied [7]. Much of this work exploits the close relation between such polynomials and expansions in Laplace–Dirichlet eigenfunctions. It may be possible to generalise this theory to q-Lidstone polynomials with the aid of polyharmonic–Dirichlet expansions.

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