Frames and numerical approximation – supplementary materials

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Abstract

This document contains supplementary materials for the paper *Frames and numerical approximation* by B. Adcock & D. Huybrechs [3].

SM1 Example 1. Fourier frames for complex geometries

Consider the frame (3.1) over a domain $\Omega \subseteq (-1,1)^d$.

SM1.1 The kernel of $\mathcal{G}$

We first characterize the kernel of the Gram operator:

Proposition SM1.1. *Let $\mathcal{G}$ be the Gram operator (2.7) of the frame (3.1). Then*

$$\mathrm{Ker}(\mathcal{G}) = \left\{ \{\hat{f}_n\}_{n\in\mathbb{Z}^d} : f \in \mathrm{L}^2(-1,1)^d,\ f(x) = 0 \text{ a.e. } x \in \Omega \right\},$$

where $\hat{f}_n = \int_{(-1,1)^d} f(t)\overline{\phi_n(t)}\,\mathrm{d}t$ are the Fourier coefficients of $f \in \mathrm{L}^2(-1,1)^d$.

Proof. Let $f \in \mathrm{L}^2(-1,1)^d$ with $f|_\Omega = 0$. Let $x = \{\hat{f}_n\}_{n\in\mathbb{Z}^d}$. Then

$$\mathcal{G}x = \{\langle f, \phi_n\rangle\}_{n\in\mathbb{Z}^d} = 0.$$

Conversely, if $x \in \mathrm{Ker}(\mathcal{G})$ then $x = \{\hat{f}_n\}_{n\in\mathbb{Z}^d}$ for some $f \in \mathrm{L}^2(-1,1)^d$. Notice that

$$0 = x^*\mathcal{G}x = \left\| \sum_{n\in\mathbb{Z}^d} x_n \phi_n \right\|^2 = \int_\Omega |f(x)|^2\,\mathrm{d}x.$$

Hence $f(x) = 0$ a.e. for $x \in \Omega$. □

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SM1.2 Proofs of Propositions 4.5 and 5.8

We first require some notation. Let $\mathrm{H}^k(\Omega)$ be the k^{th} standard Sobolev space of functions on Ω, and $\mathrm{H}^k_0(\Omega)$ be the closure of $\mathrm{C}^\infty_c(\Omega)$ with respect to the corresponding norm $\|\cdot\|_{\mathrm{H}^k(\Omega)}$. Throughout this section, we also let $c_{k,d}$ denote an arbitrary constant that is independent of N and of the particular function f being approximated.

The following lemma confirms the existence of small norm extensions:

Lemma SM1.2. *Let $\Omega \subseteq (-1,1)^d$ be a Lipschitz domain. Then for any $k \in \mathbb{N}$ there exists a linear extension operator $\mathcal{E} : \mathrm{H}^k(\Omega) \to \mathrm{H}^k_0(-1,1)^d$ satisfying $\mathcal{E}f(x) = f(x)$ a.e. for $x \in \Omega$ and*

$$\|\mathcal{E}f\|_{\mathrm{H}^k(-1,1)^d} \leq c_{k,d}\|f\|_{\mathrm{H}^k(\Omega)}. \tag{SM1.1}$$

Proof. Recall that there exists an extension operator $\mathcal{E}' : \mathrm{H}^k(\Omega) \to \mathrm{H}^k(-1,1)^d$ satisfying $\mathcal{E}f(x) = f(x)$ a.e. for $x \in \Omega$ and $\|\mathcal{E}'f\|_{\mathrm{H}^k(-1,1)^d} \leq c_{k,d}\|\mathcal{E}'f\|_{\mathrm{H}^k(\Omega)}$ some positive constant $c_{k,d}$ [1]. Moreover, there also exists a smooth bump function $g \in \mathrm{C}^\infty_0(\mathbb{R}^d)$ satisfying $g(x) = 1$, $x \in \Omega$ and $g(x) = 0$, $x \notin (-1,1)^d$. We claim that the operator $\mathcal{E}f(x) = g(x)\mathcal{E}'f(x)$ is a suitable extension operator. Certainly $\mathcal{E}f(x) = f(x)$, a.e. for $x \in \Omega$ and $\mathcal{E}f(x) = 0$ outside $(-1,1)^d$. Also $\mathcal{E}f \in \mathrm{H}^k(\Omega)$ by construction. Finally, a simple calculation confirms that (SM1.1) holds for some suitable constant $c_{k,d}$. ☐

In order to prove results about this frame, we need to recall some basic Fourier analysis. In particular, the following result is standard (see [5, Chpt. 5]):

Lemma SM1.3. *Let $g \in \mathrm{H}^{kd}_0(-1,1)^d$ and consider its Fourier coefficients $\hat{g}_n = \int_{(-1,1)^d} g(t)\overline{\phi_n(t)}\,\mathrm{d}t$, where $\phi_n(t)$ is as in (3.1). If I_N is as in (3.2) then*

$$\left\| g - \sum_{n \notin I_N} \hat{g}_n \phi_n \right\|_{L^2(-1,1)^d} \leq c_{k,d} N^{-k} \|g\|_{\mathrm{H}^{kd}(-1,1)^d}.$$

Note that it is not necessary for g to vanish on the boundary of $(-1,1)^d$ for this lemma to hold. Instead, it needs to belong to the periodic Sobolev space $\mathrm{H}^{kd}_p(-1,1)^d$ over the unit torus $(-1,1)^d$ [5]. However, all construction we consider will yield functions in $\mathrm{H}^{kd}_0(-1,1)^d$. Hence we state the result as above. With this to hand, we can now prove Proposition 5.8:

Proof of Proposition 5.8. By Lemma SM1.2 there is an extension $g \in \mathrm{H}^{kd}_0(-1,1)^d$ of f with $\|g\|_{\mathrm{H}^{kd}(-1,1)^d} \leq c_{k,d}\|f\|_{\mathrm{H}^{kd}(\Omega)}$. Let $\hat{g}_n$ be the Fourier coefficients of g on $(-1,1)^d$ and set $x_n = \hat{g}_n$ for $n \in I = \mathbb{Z}^d$. Then, by Parseval's formula for the Fourier basis on $(-1,1)^d$, we have $\|x\| = \|g\|_{\mathrm{L}^2(-1,1)^d} \leq c_{k,d}\|f\|_{\mathrm{H}^{kd}(\Omega)}$. Furthermore, $\|f - \mathcal{T}_N x\| \leq \|g - \mathcal{T}_N x\|_{\mathrm{L}^2(-1,1)^d}$. But $\mathcal{T}_N x$ is just the partial Fourier series of the function g on $(-1,1)^d$. Hence, by Lemma SM1.3

$$\|f - \mathcal{T}_N x\| \leq c_{k,d} N^{-k} \|g\|_{\mathrm{H}^{kd}(-1,1)^d} \leq c_{k,d} N^{-k} \|f\|_{\mathrm{H}^{kd}(\Omega)},$$

which gives the result. ☐

We are now able to prove Proposition 4.5:

Proof of Proposition 4.5. Define the coefficients x_n, $n \in \mathbb{Z}^d$, by $x_0 = 1$ and $x_n = 0$ otherwise. Then (4.2) gives that $B_N \geq \|\phi_0\|^2 = 2^{-d}\mathrm{Vol}(\Omega) \gtrsim 1$. It therefore suffices to consider the lower frame bound A_N. Let $g \in \mathrm{H}_0^{kd}(-1,1)^d$ be such that $\|g\|_{\mathrm{L}^2(-1,1)^d} = 1$ and $g(x) = 0$ a.e. for $x \in \Omega$. Suppose that $x \in \ell^2(I)$ is the vector of Fourier coefficients of g. Then

$$A_N \leq \frac{\|\mathcal{T}_N x\|^2}{\|x\|^2} = \frac{\|g - \mathcal{T}_N x\|^2}{\|g\|^2_{\mathrm{L}^2(-1,1)^d}} \leq c_{k,d}^2 N^{-2k} \|g\|^2_{\mathrm{H}^{kd}(-1,1)^d},$$

by Lemma SM1.3. □

SM2 Example 2. Augmented Fourier basis

We now consider the augmented Fourier frame (3.3).

SM2.1 The kernel of $\mathcal{G}$

We first characterize the kernel of the Gram operator:

Proposition SM2.1. *Let $\mathcal{G}$ be the Gram operator (2.7) of the frame (3.3). Then $\mathcal{G}$ consists of vectors of the form*

$$\begin{bmatrix} \{\langle p, \psi_k \rangle\}_{k=1}^K \\ \{\langle -p, \varphi_n \rangle\}_{n \in \mathbb{Z}} \end{bmatrix}, \qquad p \in \mathbb{P}_K^0. \tag{SM2.1}$$

In particular, $\mathrm{Ker}(\mathcal{G})$ *has dimension K.*

Proof. Let x be of the form (SM2.1). Then

$$\mathcal{T} x = \sum_{K=1}^{K} \langle p, \psi_k \rangle \psi_k + \sum_{n \in \mathbb{Z}} \langle -p, \varphi_n \rangle \varphi_n = p - p = 0.$$

Hence $\mathcal{G} x = \mathcal{T}^* \mathcal{T} x = 0$. Conversely, let $x \in \mathrm{Ker}(\mathcal{G})$. We may write $x = \begin{bmatrix} \{\langle p, \psi_k \rangle\}_{k=1}^K \\ \{\langle f, \varphi_n \rangle\}_{n \in \mathbb{Z}} \end{bmatrix}$, where $p \in \mathbb{P}_K^0$ and $f \in \mathrm{L}^2(-1,1)$. Then

$$x^* \mathcal{G} x = \|\mathcal{T} x\|^2 = \left\| \sum_{k=1}^{K} \langle p, \psi_k \rangle \psi_k + \sum_{n \in \mathbb{Z}} \langle f, \varphi_n \rangle \varphi_n \right\| = \|p + f\|.$$

Hence $f = -p$ and therefore x is of the form (SM2.1). □

SM2.2 The canonical dual frame

Although this frame is not tight, its canonical dual frame has an explicit expression. Let $\mathcal{Q}_K$ and $\mathcal{F}_{N-K}$ be the orthogonal projections onto $\mathrm{span}\{\psi_1, \ldots, K\}$ and $\mathrm{span}\left\{\varphi_n : n = -\frac{N-K}{2}, \ldots, \frac{N-K}{2} - 1\right\}$ respectively. Then we have the following:

Proposition SM2.2. *The frame operator of the frame (3.3) satisfies*

$$\mathcal{S} = \mathcal{I} + \mathcal{Q}_K, \qquad \mathcal{S}^{-1} = \mathcal{I} - \frac{1}{2} \mathcal{Q}_K. \tag{SM2.2}$$

Furthermore, the truncated canonical dual frame expansion $f_N = \sum_{n \in I_N} \langle \mathcal{S}^{-1} f, \phi_n \rangle \phi_n$ satisfies

$$f - f_N = (\mathcal{I} - \mathcal{F}_{N-K}) \, \mathcal{S}^{-1} f. \tag{SM2.3}$$

Proof. Observe that

$$\mathcal{S} = \sum_{n\in\mathbb{Z}} \langle \cdot, \varphi_n \rangle \varphi_n + \sum_{k=1}^{K} \langle \cdot, \psi_n \rangle \psi_n = \mathcal{I} + \mathcal{Q}_K,$$

and therefore

$$\left(\mathcal{I} - \frac{1}{2}\mathcal{Q}_K\right)(\mathcal{I} + \mathcal{Q}_K) = \mathcal{I} - \frac{1}{2}\mathcal{Q}_K + \mathcal{Q}_K - \frac{1}{2}\mathcal{Q}_K^2 = \mathcal{I},$$

since $\mathcal{Q}_K$ is a projection. This gives (SM2.2).

Consider (SM2.3). The truncated canonical dual frame expansion is

$$f_N = \mathcal{F}_{N-K}\mathcal{S}^{-1}f + \mathcal{Q}_K\mathcal{S}^{-1}f = \mathcal{F}_{N-K}f - \frac{1}{2}\mathcal{F}_{N-K}\mathcal{Q}_K f + \frac{1}{2}\mathcal{Q}_K f,$$

as required. ☐

The expression (SM2.3) shows that the tail of the canonical dual frame expansion is equal to the tail of the Fourier expansion of the function $\mathcal{S}^{-1}f = f - \frac{1}{2}\mathcal{Q}_K f$. In general, this cannot converge quickly as $N \to \infty$ when f is nonperiodic. Indeed, if f is nonperiodic then the function $\mathcal{S}^{-1}f$ is also nonperiodic. Nor can it be well approximated by a periodic function. In particular, for large K one has $\mathcal{S}^{-1}f \approx \frac{1}{2}f$, which is just a scaled version of f.

SM2.3 Proofs of Propositions 4.6 and 5.9

To prove these results, we shall first exploit the fact that the Fourier coefficients of a smooth but nonperiodic function f can be written as the sum of Fourier coefficients of a polynomial, plus a remainder term which is asymptotically small. This is rather standard approach in the analysis of Fourier expansions [2, 4, 6, 7].

We first require the following lemma:

Lemma SM2.3. *Let $K \in \mathbb{N}$ and suppose that $c_0, \ldots, c_{K-1} \in \mathbb{C}$. Then there exists a unique polynomial $p \in \mathbb{P}_K^0$ satisfying $p^{(r)}(1) - p^{(r)}(-1) = c_r$ for $r = 0, \ldots, K-1$.*

Proof. For $r = 0, 1, \ldots$, let $p_r(t) = \frac{2^r}{(r+1)!}B_{r+1}((t+1)/2)$, where $B_{r+1} \in \mathbb{P}_{r+1}$ is the rth Bernoulli polynomial. By properties of the Bernoulli polynomials, we have $\int_{-1}^{1} p_r(t)\,\mathrm{d}t = 0$ and $p_s^{(r)}(1) - p_s^{(r)}(-1) = \delta_{r,s}$, $r, s = 0, \ldots, K-1$. Hence, such a $p \in \mathbb{P}_K$ exists and can be written as $p(t) = \sum_{r=0}^{K-1} c_r p_{r+1}(t)$. For uniqueness, we note that the system $\{1, p_1, p_2, \ldots, p_{K-1}\}$ forms a basis for the space $\mathbb{P}_{K-1}$. ☐

Consider the Fourier coefficient $\langle f, \varphi_n \rangle$ of a function f, where $\varphi_n(t) = \frac{1}{\sqrt{2}}\mathrm{e}^{\mathrm{i}n\pi t}$. If $f \in \mathrm{H}^k(-1,1)$ and $n \neq 0$ then integrating by parts k times gives

$$\begin{aligned}\langle f, \varphi_n \rangle &= \frac{1}{\sqrt{2}} \int_{-1}^{1} f(t)\mathrm{e}^{-\mathrm{i}n\pi t}\,\mathrm{d}t \\ &= \frac{(-1)^{n+1}}{\sqrt{2}} \sum_{r=0}^{k-1} \frac{f^{(r)}(1) - f^{(r)}(-1)}{(\mathrm{i}n\pi)^{r+1}} + \frac{1}{(\mathrm{i}n\pi)^k}\langle f^{(k)}, \varphi_n \rangle.\end{aligned}$$

In particular, for the polynomials $p_r(t)$ introduced in the above proof, we have

$$\langle p_r, \varphi_n \rangle = \frac{(-1)^{n+1}}{\sqrt{2}(\mathrm{i}n\pi)^{r+1}}, \quad \langle p_r, \varphi_0 \rangle = 0 \qquad \text{(SM2.4)}$$

Hence, if $f \in \mathrm{H}^K(-1,1)$ and $p \in \mathbb{P}_K$ is the polynomial with

$$p^{(r)}(1) - p^{(r)}(-1) = f^{(r)}(1) - f^{(r)}(-1), \quad r = 0, \ldots, K-1,$$

then we have

$$\langle f, \varphi_n \rangle = \langle p, \varphi_n \rangle + \frac{1}{(\mathrm{i}n\pi)^k} \langle f^{(k)}, \varphi_n \rangle. \tag{SM2.5}$$

We may now prove Proposition 5.9:

Proof of Proposition 5.9. Since f has $k-1$ continuous derivatives, Lemma SM2.3 ensures the existence of a polynomial $p \in \mathbb{P}^0_k$ with $p^{(r)}(1) - p^{(r)}(-1) = f^{(r)}(1) - f^{(r)}(-1)$ for $r = 0, \ldots, k-1$. Write $p = \sum_{r=1}^{k} a_r \psi_r$ and define the coefficients $x \in \ell^2(I)$ so that

$$\mathcal{T} x = \sum_{n \in \mathbb{Z}} \langle f - p, \varphi_n \rangle \varphi_n + \sum_{k=1}^{K} a_k \psi_k,$$

Note that

$$\|x\|^2 = \sum_{n \in \mathbb{Z}} |\langle f - p, \varphi_n \rangle|^2 + \|a\|^2 = \|f - p\|^2 + \|a\|^2.$$

It is straightforward to see that there exists a constant c_k such that

$$\|a\|, \|p\| \leq c_k \max_{r=0,\ldots,k-1} |f^{(r)}(1) - f^{(r)}(-1)| \leq \sqrt{2} c_k \|f\|_{\mathrm{H}^k},$$

where in the last step we use the fact that $f^{(r)}(1) - f^{(r)}(-1) = \int_{-1}^{1} f^{(r+1)}(t) \,\mathrm{d}t$. Moreover, by Parseval's theorem and (SM2.5),

$$\begin{aligned} \|f - \mathcal{T}_N x\|^2 &= \left\| f - p - \sum_{n=-\frac{N-K}{2}}^{\frac{N-K}{2}-1} \langle f - p, \varphi_n \rangle \phi_n \right\|^2 \\ &\leq \sum_{|n| \geq \frac{N-K}{2}} |\langle f - p, \varphi_n \rangle|^2 \\ &\leq \frac{1}{((N-K)\pi/2)^{2k}} \|f^{(k)}\|^2. \end{aligned}$$

This now gives the result. $\square$

Finally, give the proof of Proposition 4.6:

Proof of Proposition 4.6. Since the frame Φ contains an orthonormal basis, we have $B_N \geq 1$. Conversely, let $p = p_{K-1}$ be as in the proof of Proposition SM2.3. Write $p = \sum_{k=1}^{K} z_k \psi_k$ for some coefficients z_k, and let $y_n = -\langle p, \varphi_n \rangle$, $n \in \mathbb{Z}$. Then

$$A_N = \min_{x \in \mathbb{C}^N \, \|x\|=1} x^* G_N x \leq \frac{\left\| p - \sum_{n=-\frac{N-K}{2}}^{\frac{N-K}{2}-1} \langle p, \varphi_n \rangle \phi_n \right\|^2}{\sum_{n=-\frac{N-K}{2}}^{\frac{N-K}{2}-1} |\langle p, \varphi_n \rangle|^2} \leq \frac{E_N}{\|p\|^2 - E_N},$$

where $E_N = \sum_{|n| \geq \frac{N-K}{2}} |\langle p, \varphi_n \rangle|^2$. Using (SM2.4), we deduce that $E_N \lesssim N^{1-2K}$ for $N > K$. $\square$

SM3 Example 3. Polynomials plus modified polynomials

We now consider the frame (3.6).

SM3.1 The kernel of $\mathcal{G}$

Proposition SM3.1. *Let $\mathcal{G}$ be the Gram operator (2.7) of the frame (3.6). Then* $\mathrm{Ker}(\mathcal{G})$ *consists of vectors of the form*

$$\begin{bmatrix} \{\langle f, \varphi_n \rangle\}_{n\in\mathbb{N}} \\ -c \end{bmatrix}, \tag{SM3.1}$$

where $f \in \mathrm{L}^2(-1,1)$ is arbitrary and $c \in \ell^2(\mathbb{N})$ is any vector satisfying $\sum_{n\in\mathbb{N}} c_n \psi_n = f$, where $\psi_n = w\varphi_n$.

Proof. Let x be of the form (SM3.1). Then

$$\mathcal{T}x = \sum_{n\in\mathbb{N}} \langle f, \varphi_n \rangle \varphi_n - \sum_{n\in\mathbb{N}} c_n \psi_n = f - f = 0.$$

Hence $\mathcal{G}x = \mathcal{T}^*\mathcal{T}x = 0$. Conversely, let $x \in \mathrm{Ker}(\mathcal{G})$ and write $x = \begin{bmatrix} \{\langle f, \varphi_n \rangle\}_{n\in\mathbb{N}} \\ -c \end{bmatrix}$ for some $f \in \mathrm{L}^2(-1,1)$ and $c \in \ell^2(\mathbb{N})$. Then

$$x^*\mathcal{G}x = \|\mathcal{T}x\|^2 = \left\| f - \sum_{n\in\mathbb{N}} c_n \psi_n \right\|.$$

Hence c satisfies $\sum_{n\in\mathbb{N}} c_n \psi_n = f$. ☐

SM3.2 The canonical dual frame

As with the previous example, we can give an explicit expression for the canonical dual frame in this case. To this end, let $\mathcal{Q}_N$ be the orthogonal projection onto $\mathrm{span}\{\varphi_1, \ldots, \varphi_{N/2}\} = \mathbb{P}_{N/2-1}$. Then

Proposition SM3.2. *The frame operator of the frame (3.6) satisfies*

$$\mathcal{S}f = v^{-1} f, \qquad \mathcal{S}^{-1} = vf, \tag{SM3.2}$$

where $v = (1+w^2)^{-1}$. Furthermore, the truncated canonical dual frame expansion $f_N = \sum_{n\in I_N} \langle \mathcal{S}^{-1} f, \phi_n \rangle \phi_n$ satisfies

$$f - f_N = (\mathcal{I} - \mathcal{Q}_N)(vf) + w\,(\mathcal{I} - \mathcal{Q}_N)\,(wvf). \tag{SM3.3}$$

Proof. Notice that if f is in $\mathrm{L}^2(-1,1)$ then so is the function wf, since $w \in \mathrm{L}^\infty(-1,1)$. Hence

$$\mathcal{S}f = \sum_{n=1}^{\infty} \langle f, \varphi_n \rangle \varphi_n + w \sum_{n=1}^{\infty} \langle f, w\varphi_n \rangle \varphi_n = \left(1 + w^2\right) f,$$

which immediately gives (SM3.2). Therefore

$$f_N = \sum_{n=1}^{N} \langle \mathcal{S}^{-1} f, \varphi_n \rangle \varphi_n + w \sum_{n=1}^{N} \langle \mathcal{S}^{-1} f, w\varphi_n \rangle \varphi_n = \mathcal{Q}_N(vf) + w\mathcal{Q}_N(wvf).$$

This gives

$$f - f_N = vf + w^2 vf - \mathcal{Q}_N(vf) - w\mathcal{Q}_N(wvf) = (\mathcal{I} - \mathcal{Q}_N)(vf) + w\left(\mathcal{I} - \mathcal{Q}_N\right)(wvf),$$

as required. □

Note that the functions vf and wvf are both nondifferentiable at $t = -1$ when $w(t)$ is given by (3.5). Hence the convergence rate of the projections $\mathcal{Q}_N(vf)$ and $\mathcal{Q}_N(wvf)$ onto the space $\mathbb{P}_{N/2-1}$ is algebraic in N with small index.

SM3.3 Proofs of Propositions 4.7 and 5.10

Proof of Proposition 4.7. Since the frame Φ contains an orthonormal basis, we have $B_N \geq 1$. Moreover,

$$A_N = \min_{x \in \mathbb{C}^N \, \|x\|=1} x^* G_N x = \min_{\substack{p,q \in \mathbb{P}_{N/2-1} \\ \|p\|+\|q\| \neq 0}} \frac{\|p + wq\|^2}{\|p\|^2 + \|q\|^2}.$$

Set $q(t) = (1+t)^{N/2-1} \in \mathbb{P}_{N/2-1}$ and write $r(t) = w(t)q(t)$. Let $p(t) = \sum_{n=1}^{N/2} \langle r, \varphi_n \rangle \varphi_n$ be the orthogonal projection of r in the Legendre polynomial basis. Since $\|q\|^2 = \frac{2^{N-1}}{N-1}$ and $\|r\|^2 = \frac{2^{N+2\alpha-1}}{N+2\alpha-1}$ we have

$$A_N \leq \frac{\|r - p\|^2}{\|r - p\|^2 + \frac{2^{N-1}}{N-1} + \frac{2^{N+2\alpha-1}}{N+2\alpha-1}}. \tag{SM3.4}$$

We now examine $\|r - p\|^2$. Using [5, (5.4.13)] we deduce that

$$\begin{aligned} \|r - p\|^2 &\leq \frac{\Gamma(N/2 - k)}{\Gamma(N/2 + k)} \int_{-1}^{1} |r^{(k)}(t)|^2 (1 - t^2)^k \, \mathrm{d}t \\ &= \frac{\Gamma(N/2 - k)}{\Gamma(N/2 + k)} \left(\frac{\Gamma(N/2 + \alpha)}{\Gamma(N/2 + \alpha - k)} \right)^2 \int_{-1}^{1} (1+t)^{N+2\alpha-2k-2} (1 - t^2)^k \, \mathrm{d}t, \end{aligned}$$

for $0 \leq k \leq N/2$. Setting $k = N/2$ and noting that $\int_{-1}^{1} (1+t)^{2\alpha-2}(1-t^2)^{N/2} \, \mathrm{d}t \leq 1$ for $N \geq 4$ now gives

$$\|r - p\|^2 \leq \frac{1}{\Gamma(N)} \left(\frac{\Gamma(N/2 + \alpha)}{\Gamma(\alpha)} \right)^2.$$

Stirling's formula now gives

$$\frac{1}{\Gamma(N)} \left(\frac{\Gamma(N/2 + \alpha)}{\Gamma(\alpha)} \right)^2 \sim \frac{2\sqrt{\pi}}{\Gamma(\alpha)^2 (2\mathrm{e})^\alpha} N^{2\alpha - 1/2} 2^{-N}, \quad N \to \infty.$$

The result now follows immediately from (SM3.4). □

Proof of Proposition 5.10. The proof is straightforward. If $f(t) = w(t)g(t) + h(t)$, then we let $w \in \ell^2(I)$ be the infinite vector of coefficients of g and h with respect to the orthonormal basis $\{\varphi_n\}_{n \in \mathbb{N}}$. Note that this vector satisfies

$$\|w\|_2 = \sqrt{\|g\|^2 + \|h\|^2} \leq \|g\| + \|h\|,$$

by Parseval's equality. Also

$$f(t) - \mathcal{T}_N w(t) = f(t) - w(t) \sum_{n=1}^{N/2} \langle g, \varphi_n \rangle \varphi_n(t) - \sum_{n=1}^{N/2} \langle h, \varphi_n \rangle \varphi_n(t)$$
$$= g(t) - \mathcal{Q}_N g(t) + w(t) \left(h(t) - \mathcal{Q}_N h(t) \right),$$

and therefore

$$\|f - \mathcal{T}_N w\| \leq \|g - \mathcal{Q}_N g\| + \|w(h - \mathcal{Q}_N h)\| \leq \|g - \mathcal{Q}_N g\| + w_{\max} \|h - \mathcal{Q}_N h\|,$$

as required. □

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