Frames and numerical approximation – supplementary materials

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December 14, 2016

Abstract

This document contains supplementary materials for the paper Frames and numerical approximation by B. Adcock & D. Huybrechs [3].

SM1 Example 1. Fourier frames for complex geometries

Consider the frame (3.1) over a domain Ω ⊆ (−1, 1)d.

SM1.1 The kernel of 𝐇

We first characterize the kernel of the Gram operator:

Proposition SM1.1. Let 𝐇 be the Gram operator (2.7) of the frame (3.1). Then

\[ \text{Ker}(H) = \left\{ \{\hat{f}_n\}_{n \in \mathbb{Z}^d} : f \in L^2(-1, 1)^d, \ f(x) = 0 \ a.e. \ x \in \Omega \right\}, \]

where \( \hat{f}_n = \int_{(-1,1)^d} f(t) \overline{\phi_n(t)} \, dt \) are the Fourier coefficients of \( f \in L^2(-1, 1)^d \).

Proof. Let \( f \in L^2(-1, 1)^d \) with \( f|_{\Omega} = 0 \). Let \( x = \{\hat{f}_n\}_{n \in \mathbb{Z}^d} \). Then

\[ Hx = \{\langle f, \phi_n \rangle\}_{n \in \mathbb{Z}^d} = 0. \]

Conversely, if \( x \in \text{Ker}(H) \) then \( x = \{\hat{f}_n\}_{n \in \mathbb{Z}^d} \) for some \( f \in L^2(-1, 1)^d \). Notice that

\[ 0 = x^* Hx = \left\| \sum_{n \in \mathbb{Z}^d} x_n \phi_n \right\|^2 = \int_{\Omega} |f(x)|^2 \, dx. \]

Hence \( f(x) = 0 \ a.e. \ for \ x \in \Omega. \)

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SM1.2 Proofs of Propositions 4.5 and 5.8

We first require some notation. Let \( H^k(\Omega) \) be the \( k \)-th standard Sobolev space of functions on \( \Omega \), and \( H^k_0(\Omega) \) be the closure of \( C_0^\infty(\Omega) \) with respect to the corresponding norm \( \| \cdot \|_{H^k(\Omega)} \). Throughout this section, we also let \( c_{k,d} \) denote an arbitrary constant that is independent of \( N \) and of the particular function \( f \) being approximated.

The following lemma confirms the existence of small norm extensions:

Lemma SM1.2. Let \( \Omega \subseteq (-1,1)^d \) be a Lipschitz domain. Then for any \( k \in \mathbb{N} \) there exists a linear extension operator \( E : H^k(\Omega) \to H^k_0(-1,1)^d \) satisfying \(Ef(x) = f(x)\) a.e. for \( x \in \Omega \) and

\[
\|Ef\|_{H^k_0(-1,1)^d} \leq c_{k,d}\|f\|_{H^k(\Omega)}. \tag{SM1.1}
\]

Proof. Recall that there exists an extension operator \( E' : H^k(\Omega) \to H^k(-1,1)^d \) satisfying \(E'f(x) = f(x)\) a.e. for \( x \in \Omega \) and \( \|E'f\|_{H^k(-1,1)^d} \leq c_{k,d}\|f\|_{H^k(\Omega)} \) some positive constant \( c_{k,d} \) [1]. Moreover, there also exists a smooth bump function \( g \in C_0^{\infty}(\mathbb{R}^d) \) satisfying \( g(x) = 1, x \in \Omega \) and \( g(x) = 0, x \notin (-1,1)^d \). We claim that the operator \( Ef(x) = g(x)E'f(x) \) is a suitable extension operator. Certainly \(Ef(x) = f(x)\), a.e. for \( x \in \Omega \) and \( Ef(x) = 0 \) outside \((-1,1)^d\). Also \( Ef \in H^k(\Omega) \) by construction. Finally, a simple calculation confirms that (SM1.1) holds for some suitable constant \( c_{k,d} \).

In order to prove results about this frame, we need to recall some basic Fourier analysis. In particular, the following result is standard (see [5, Chpt. 5]):

Lemma SM1.3. Let \( g \in H^k_0(-1,1)^d \) and consider its Fourier coefficients \( \hat{g}_n = \int_{(-1,1)^d} g(t)\overline{\phi_n(t)} \, dt \), where \( \phi_n(t) \) is as in (3.1). If \( I_N \) is as in (3.2) then

\[
\left\| g - \sum_{n \notin I_N} \hat{g}_n\phi_n \right\|_{L^2(-1,1)^d} \leq c_{k,d}N^{-k}\|g\|_{H^k_0(-1,1)^d}.
\]

Note that it is not necessary for \( g \) to vanish on the boundary of \((-1,1)^d\) for this lemma to hold. Instead, it needs to belong to the periodic Sobolev space \( H^k_0(-1,1)^d \) over the unit torus \((-1,1)^d\) [5]. However, all construction we consider will yield functions in \( H^k_0(-1,1)^d \). Hence we state the result as above. With this to hand, we can now prove Proposition 5.8:

Proof of Proposition 5.8. By Lemma SM1.2 there is an extension \( g \in H^k_0(-1,1)^d \) of \( f \) with \( \|g\|_{H^k_0(-1,1)^d} \leq c_{k,d}\|f\|_{H^k(\Omega)} \). Let \( \hat{g}_n \) be the Fourier coefficients of \( g \) on \((-1,1)^d\) and set \( x_n = \hat{g}_n \) for \( n \in I = \mathbb{Z}^d \).

Then, by Parseval’s formula for the Fourier basis on \((-1,1)^d\), we have \( \|x\| = \|g\|_{L^2(-1,1)^d} \leq c_{k,d}\|f\|_{H^k(\Omega)} \). Furthermore, \( \|f - \mathcal{T}_N x\| \leq \|g - \mathcal{T}_N x\|_{L^2(-1,1)^d} \). But \( \mathcal{T}_N x \) is just the partial Fourier series of the function \( g \) on \((-1,1)^d\). Hence, by Lemma SM1.3

\[
\|f - \mathcal{T}_N x\| \leq c_{k,d}N^{-k}\|g\|_{H^k_0(-1,1)^d} \leq c_{k,d}N^{-k}\|f\|_{H^k(\Omega)},
\]

which gives the result.

We are now able to prove Proposition 4.5:

Proof of Proposition 4.5. Define the coefficients \( x_n, n \in \mathbb{Z}^d \), by \( x_0 = 1 \) and \( x_n = 0 \) otherwise. Then (4.2) gives that \( B_N \geq \|\phi_0\|^2 = 2^{-d}\text{Vol}(\Omega) \geq 1 \). It therefore suffices to consider the lower frame.
bound $A_N$. Let $g \in H^k_d(-1,1)^d$ be such that $\|g\|_{L^2(-1,1)^d} = 1$ and $g(x) = 0$ a.e. for $x \in \Omega$. Suppose that $x \in \ell^2(I)$ is the vector of Fourier coefficients of $g$. Then

$$A_N \leq \frac{\|T_N x\|^2}{\|x\|^2} = \frac{\|g - T_N x\|^2}{\|g\|_{L^2(-1,1)^d}^2} \leq c^2_{k,d} N^{-2k} \|g\|^2_{H^k_d(-1,1)^d},$$

by Lemma SM1.3.

**SM2 Example 2. Augmented Fourier basis**

We now consider the augmented Fourier frame (3.3).

**SM2.1 The kernel of $\mathcal{G}$**

We first characterize the kernel of the Gram operator:

**Proposition SM2.1.** Let $\mathcal{G}$ be the Gram operator (2.7) of the frame (3.3). Then $\mathcal{G}$ consists of vectors of the form

$$\left\{ \{\langle p, \psi_k \rangle\}_{k=1}^K, \{\langle -p, \varphi_n \rangle\}_{n \in \mathbb{Z}} \right\}, \quad p \in \mathbb{P}_K^0. \tag{SM2.1}$$

In particular, Ker($\mathcal{G}$) has dimension $K$.

**Proof.** Let $x$ be of the form (SM2.1). Then

$$T x = \sum_{K=1}^K \langle p, \psi_k \rangle \psi_k + \sum_{n \in \mathbb{Z}} \langle -p, \varphi_n \rangle \varphi_n = p - p = 0.$$

Hence $\mathcal{G} x = T^* T x = 0$. Conversely, let $x \in$ Ker($\mathcal{G}$). We may write $x = \left\{ \{\langle p, \psi_k \rangle\}_{k=1}^K, \{\langle f, \varphi_n \rangle\}_{n \in \mathbb{Z}} \right\}$, where $p \in \mathbb{P}_K^0$ and $f \in L^2(-1,1)$. Then

$$x^* \mathcal{G} x = \|T x\|^2 = \left\| \sum_{k=1}^K \langle p, \psi_k \rangle \psi_k + \sum_{n \in \mathbb{Z}} \langle f, \varphi_n \rangle \varphi_n \right\| = \|p + f\|.$$

Hence $f = -p$ and therefore $x$ is of the form (SM2.1). □

**SM2.2 The canonical dual frame**

Although this frame is not tight, its canonical dual frame has an explicit expression. Let $Q_K$ and $\mathcal{F}_{N-K}$ be the orthogonal projections onto span{$\psi_1, \ldots, K$} and span {$\varphi_n : n = -\frac{N-K}{2}, \ldots, -\frac{N-K}{2} - 1$} respectively. Then we have the following:

**Proposition SM2.2.** The frame operator of the frame (3.3) satisfies

$$S = I + Q_K, \quad S^{-1} = I - \frac{1}{2} Q_K. \tag{SM2.2}$$

Furthermore, the truncated canonical dual frame expansion $f_N = \sum_{n \in I_N} \langle S^{-1} f, \varphi_n \rangle \varphi_n$ satisfies

$$f - f_N = (I - \mathcal{F}_{N-K}) S^{-1} f. \tag{SM2.3}$$
Lemma SM2.3. Let $F_0, F_2, F_4, F_6, F_7$ be the Fourier expansions of a smooth but nonperiodic function $f$. The remainder term which is asymptotically small. This is a rather standard approach in the analysis of Fourier expansions. To prove these results, we shall first exploit the fact that the Fourier coefficients of a smooth but nonperiodic function $f$ can be written as the sum of Fourier coefficients of a polynomial, plus a nonperiodic term. Nor can it be well approximated by a periodic function. In particular, for large $K$ one has $S^{-1}f = f - \frac{1}{2} Q_K f$. This is just a scaled version of $f$.

Proof. Observe that

$$ S = \sum_{n \in Z} \langle \cdot, \varphi_n \rangle \varphi_n + \sum_{k=1}^{K} \langle \cdot, \psi_n \rangle \psi_n = I + Q_K, $$

and therefore

$$ \left( I - \frac{1}{2} Q_K \right) (I + Q_K) = I - \frac{1}{2} Q_K + Q_K - \frac{1}{2} Q_K^2 = I, $$

since $Q_K$ is a projection. This gives (SM2.2).

Consider (SM2.3). The truncated canonical dual frame expansion is

$$ f_N = F_{N-K} S^{-1} f + Q_K S^{-1} f = F_{N-K} f - \frac{1}{2} F_{N-K} Q_K f + \frac{1}{2} Q_K f, $$

as required.

The expression (SM2.3) shows that the tail of the canonical dual frame expansion is equal to the tail of the Fourier expansion of the function $S^{-1}f = f - \frac{1}{2} Q_K f$. In general, this cannot converge quickly as $N \to \infty$ when $f$ is nonperiodic. Indeed, if $f$ is nonperiodic then the function $S^{-1}f$ is also nonperiodic. Nor can it be well approximated by a periodic function. In particular, for large $K$ one has $S^{-1}f \approx \frac{1}{2} f$, which is just a scaled version of $f$.

**SM2.3 Proofs of Propositions 4.6 and 5.9**

To prove these results, we shall first exploit the fact that the Fourier coefficients of a smooth but nonperiodic function $f$ can be written as the sum of Fourier coefficients of a polynomial, plus a remainder term which is asymptotically small. This is rather standard approach in the analysis of Fourier expansions [2, 4, 6, 7].

We first require the following lemma:

**Lemma SM2.3.** Let $K \in N$ and suppose that $c_0, \ldots, c_{K-1} \in \mathbb{C}$. Then there exists a unique polynomial $p \in \mathbb{P}_K^0$ satisfying $p^{(r)}(1) - p^{(r)}(-1) = c_r$ for $r = 0, \ldots, K - 1$.

**Proof.** For $r = 0, 1, \ldots$, let $p_r(t) = \frac{2^r}{(r+1)!} B_{r+1}(t + 1)/2$, where $B_{r+1} \in \mathbb{P}_{r+1}$ is the $r$th Bernoulli polynomial. By properties of the Bernoulli polynomials, we have $\int_{-1}^{1} p_r(t) dt = 0$ and $p_r^{(r)}(1) - p_s^{(r)}(-1) = \delta_{r,s}$, $r, s = 0, \ldots, K - 1$. Hence, such a $p \in \mathbb{P}_K$ exists and can be written as $p(t) = \sum_{r=0}^{K-1} c_r p_{r+1}(t)$. For uniqueness, we note that the system $\{1, p_1, p_2, \ldots, p_{K-1}\}$ forms a basis for the space $\mathbb{P}_{K-1}$. 

Consider the Fourier coefficient $\langle f, \varphi_n \rangle$ of a function $f$, where $\varphi_n(t) = \frac{1}{\sqrt{2}} e^{int}$. If $f \in H^k(-1, 1)$ and $n \neq 0$ then integrating by parts $k$ times gives

$$ \langle f, \varphi_n \rangle = \frac{1}{\sqrt{2}} \int_{-1}^{1} f(t) e^{-int} dt $$

$$ = (-1)^{n+1} \frac{k}{\sqrt{2}} \sum_{r=0}^{k-1} \frac{f^{(r)}(1) - f^{(r)}(-1)}{(in\pi)^{r+1}} + \frac{1}{(in\pi)^k} \langle f^{(k)}, \varphi_n \rangle. $$

In particular, for the polynomials $p_r(t)$ introduced in the above proof, we have

$$ \langle p_r, \varphi_n \rangle = \frac{(-1)^{n+1}}{\sqrt{2}(in\pi)^{r+1}}, \quad \langle p_r, \varphi_0 \rangle = 0 \quad \text{(SM2.4)} $$
Hence, if \( f \in H^K(-1, 1) \) and \( p \in \mathbb{P}_K \) is the polynomial with
\[
p^{(r)}(1) - p^{(r)}(-1) = f^{(r)}(1) - f^{(r)}(-1), \quad r = 0, \ldots, K-1,
\]
then we have
\[
\langle f, \varphi_n \rangle = \langle p, \varphi_n \rangle + \frac{1}{(2n\pi)^k} \langle f^{(k)}, \varphi_n \rangle. \tag{SM2.5}
\]

We may now prove Proposition 5.9:

**Proof of Proposition 5.9.** Since \( f \) has \( k-1 \) continuous derivatives, Lemma SM2.3 ensures the existence of a polynomial \( p \in \mathbb{P}_k^0 \) with \( p^{(r)}(1) - p^{(r)}(-1) = f^{(r)}(1) - f^{(r)}(-1) \) for \( r = 0, \ldots, k-1 \). Write \( p = \sum a_r \psi_r \) and define the coefficients \( c_r \) such that
\[
\|f - p\| \leq c_k \max_{r=0,\ldots,k-1} |f^{(r)}(1) - f^{(r)}(-1)| \leq \sqrt{2} c_k \|f\|_{H^K},
\]
where in the last step we use the fact that \( f^{(r)}(1) - f^{(r)}(-1) = \int_{-1}^1 f^{(r+1)}(t) \, dt \). Moreover, by Parseval’s theorem and (SM2.5),
\[
\|f - \mathcal{T}_N x\|^2 = \left\| f - p - \sum_{n=\frac{-N-K}{2}}^{\frac{N-K}{2}} \langle f - p, \varphi_n \rangle \varphi_n \right\|^2 \\
\leq \sum_{|n| \geq \frac{N-K}{2}} |\langle f - p, \varphi_n \rangle|^2 \\
\leq \frac{1}{((N-K)\pi/2)^{2k}} \|f^{(k)}\|^2.
\]
This now gives the result.

Finally, give the proof of Proposition 4.6:

**Proof of Proposition 4.6.** Since the frame \( \Phi \) contains an orthonormal basis, we have \( B_N \geq 1 \). Conversely, let \( p = p_{K-1} \) be as in the proof of Proposition SM2.3. Write \( p = \sum_{k=1}^K z_k \psi_k \) for some coefficients \( z_k \), and let \( y_n = \langle p, \varphi_n \rangle, \, n \in \mathbb{Z} \). Then
\[
A_N = \min_{x \in \mathbb{C}^N} x^* G_N x \leq \frac{\|p - \sum_{n=\frac{-N-K}{2}}^{\frac{N-K}{2}} \langle p, \varphi_n \rangle \varphi_n \|^2}{\sum_{n=\frac{-N-K}{2}}^{\frac{N-K}{2}} |\langle p, \varphi_n \rangle|^2} \leq \frac{E_N}{\|p\|^2 - E_N},
\]
where \( E_N = \sum_{|n| \geq \frac{N-K}{2}} |\langle p, \varphi_n \rangle|^2 \). Using (SM2.4), we deduce that \( E_N \leq N^{1-2K} \) for \( N > K \).
SM3  Example 3. Polynomials plus modified polynomials

We now consider the frame (3.6).

SM3.1  The kernel of $\mathcal{G}$

Proposition SM3.1. Let $\mathcal{G}$ be the Gram operator (2.7) of the frame (3.6). Then $\ker(\mathcal{G})$ consists of vectors of the form

$$\left\{ \langle f, \varphi_n \rangle \right\}_{n \in \mathbb{N}} - c,$$

where $f \in L^2(-1,1)$ is arbitrary and $c \in \ell^2(\mathbb{N})$ is any vector satisfying $\sum_{n \in \mathbb{N}} c_n \psi_n = f$, where $\psi_n = w \varphi_n$.

Proof. Let $x$ be of the form (SM3.1). Then

$$T x = \sum_{n \in \mathbb{N}} \langle f, \varphi_n \rangle \varphi_n - \sum_{n \in \mathbb{N}} c_n \psi_n = f - f = 0.$$\

Hence $\mathcal{G} x = T^* T x = 0$. Conversely, let $x \in \ker(\mathcal{G})$ and write $x = \left\{ \langle f, \varphi_n \rangle \right\}_{n \in \mathbb{N}}$ for some $f \in L^2(-1,1)$ and $c \in \ell^2(\mathbb{N})$. Then

$$x^* \mathcal{G} x = \|Tx\|^2 = \left\| f - \sum_{n \in \mathbb{N}} c_n \psi_n \right\|.$$\

Hence $c$ satisfies $\sum_{n \in \mathbb{N}} c_n \psi_n = f$. \hfill $\Box$

SM3.2  The canonical dual frame

As with the previous example, we can give an explicit expression for the canonical dual frame in this case. To this end, let $\mathcal{Q}_N$ be the orthogonal projection onto span$\{\varphi_1, \ldots, \varphi_{N/2}\} = \mathbb{P}_{N/2-1}$. Then

Proposition SM3.2. The frame operator of the frame (3.6) satisfies

$$S f = v^{-1} f, \quad S^{-1} = v f,$$

where $v = (1 + w^2)^{-1}$. Furthermore, the truncated canonical dual frame expansion $f_N = \sum_{n \in I_N} \langle S^{-1} f, \phi_n \rangle \phi_n$ satisfies

$$f - f_N = (I - \mathcal{Q}_N)(vf) + w (I - \mathcal{Q}_N)(wvf).$$

Proof. Notice that if $f$ is in $L^2(-1,1)$ then so is the function $w f$, since $w \in L^\infty(-1,1)$. Hence

$$S f = \sum_{n=1}^{\infty} \langle f, \varphi_n \rangle \varphi_n + w \sum_{n=1}^{\infty} \langle f, w \varphi_n \rangle \varphi_n = (1 + w^2) f,$$

which immediately gives (SM3.2). Therefore

$$f_N = \sum_{n=1}^{N} \langle S^{-1} f, \varphi_n \rangle \varphi_n + w \sum_{n=1}^{N} \langle S^{-1} f, w \varphi_n \rangle \varphi_n = \mathcal{Q}_N(vf) + w \mathcal{Q}_N(wvf).$$

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This gives
\[ f - f_N = vf + w^2vf - Q_N(vf) - wQ_N(wvf) = (I - Q_N)(vf) + w(I - Q_N)(wvf), \]
as required. \(\square\)

Note that the functions \(vf\) and \(wvf\) are both nondifferentiable at \(t = -1\) when \(w(t)\) is given by (3.5). Hence the convergence rate of the projections \(Q_N(vf)\) and \(Q_N(wvf)\) onto the space \(\mathbb{P}_{N/2-1}\) is algebraic in \(N\) with small index.

### SM3.3 Proofs of Propositions 4.7 and 5.10

**Proof of Proposition 4.7.** Since the frame \(\Phi\) contains an orthonormal basis, we have \(B_N \geq 1\). Moreover,
\[
A_N = \min_{x \in \mathbb{C}^N} \|x^*G_Nx\| = \min_{p,q \in \mathbb{P}_{N/2-1}} \frac{\|p + wq\|^2}{\|p\|^2 + \|q\|^2}.
\]
Set \(q(t) = (1 + t)^{N/2-1} \in \mathbb{P}_{N/2-1}\) and write \(r(t) = w(t)q(t)\). Let \(p(t) = \sum_{n=1}^{N/2} \langle r, \varphi_n \rangle \varphi_n\) be the orthogonal projection of \(r\) in the Legendre polynomial basis. Since \(\|q\|^2 = \frac{2^{N-1}}{N-1}\) and \(\|r\|^2 = \frac{2^{N+2a-1}}{N+2a-1}\), we have
\[
A_N \leq \frac{\|r - p\|^2}{\|r - p\|^2 + \frac{2^{N-1}}{N-1} + \frac{2^{N+2a-1}}{N+2a-1}}. \tag{SM3.4}
\]
We now examine \(\|r - p\|^2\). Using [5, (5.4.13)] we deduce that
\[
\|r - p\|^2 \leq \frac{\Gamma(N/2 - k)}{\Gamma(N/2 + k)} \int_{-1}^{1} |r^{(k)}(t)|^2 (1 - t^2)dt
\]
\[
= \frac{\Gamma(N/2 - k)}{\Gamma(N/2 + k)} \left( \frac{\Gamma(N/2 + \alpha)}{\Gamma(N/2 + \alpha - k)} \right)^2 \int_{-1}^{1} (1 + t)^{N+2a-2k-2} (1 - t^2)^k dt,
\]
for \(0 \leq k \leq N/2\). Setting \(k = N/2\) and noting that \(\int_{-1}^{1} (1 + t)^{2\alpha-2} (1 - t^2)^{N/2} dt \leq 1\) for \(N \geq 4\) now gives
\[
\|r - p\|^2 \leq \frac{1}{\Gamma(N)} \left( \frac{\Gamma(N/2 + \alpha)}{\Gamma(N/2 + \alpha)} \right)^2.
\]
Stirling’s formula now gives
\[
\frac{1}{\Gamma(N)} \left( \frac{\Gamma(N/2 + \alpha)}{\Gamma(N/2 + \alpha)} \right)^2 \sim \frac{2^{\sqrt{\pi} N^{2\alpha-1/2} 2^{-N}}}{\Gamma(\alpha)(2e)^{\alpha \alpha}}, \quad N \to \infty.
\]
The result now follows immediately from (SM3.4). \(\square\)

**Proof of Proposition 5.10.** The proof is straightforward. If \(f(t) = w(t)g(t) + h(t)\), then we let \(w \in l^2(I)\) be the infinite vector of coefficients of \(g\) and \(h\) with respect to the orthonormal basis \(\{\varphi_n\}_{n \in \mathbb{N}}\). Note that this vector satisfies
\[
\|w\|_2 = \sqrt{\|g\|^2 + \|h\|^2} \leq \|g\| + \|h\|,
\]
by Parseval’s equality. Also
\[
f(t) - T_N w(t) = f(t) - w(t) \sum_{n=1}^{N/2} \langle g, \varphi_n \rangle \varphi_n(t) - \sum_{n=1}^{N/2} \langle h, \varphi_n \rangle \varphi_n(t)
\]
\[
= g(t) - Q_N g(t) + w(t)(h(t) - Q_N h(t)),
\]
and therefore
\[
\|f - T_N w\| \leq \|g - Q_N g\| + \|w(h - Q_N h)\| \leq \|g - Q_N g\| + w_{max}\|h - Q_N h\|,
\]
as required.

References


